1/10/2018

Chapter I Diffusion equation or heat equation

Heat equation

$$\frac{\partial u}{\partial t} - \nabla^2 u = 0 \qquad \vec{x} \in D \subset \mathbb{R}^n, \; t > 0$$

domain

u: temperature

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z}$$

Laplacian in $\mathbb{R}^3$

needs

BCs $\quad \alpha(\vec{x})\, u + \beta(\vec{x}) \dfrac{\partial u}{\partial n} = \gamma(x)$

BC for $\vec{x} \in \partial D$

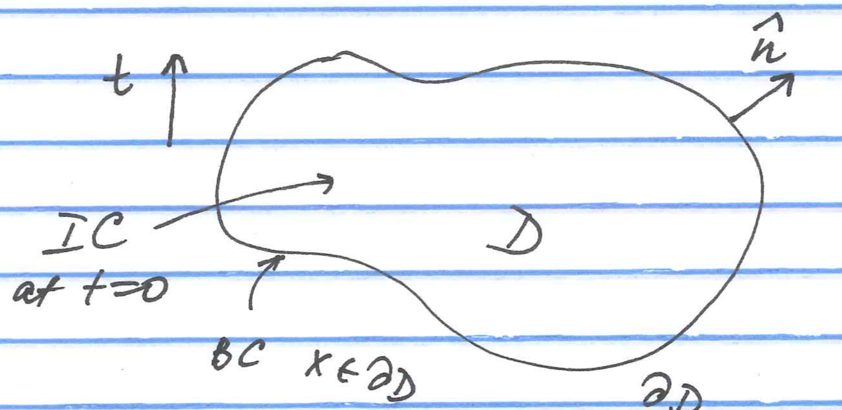
boundary of D

α, β, γ: given

$\dfrac{\partial u}{\partial n} = \nabla u \cdot \hat{n}$: normal derivative

$\hat{n}$: outward unit normal vector

We are studying heat conduction in domain D



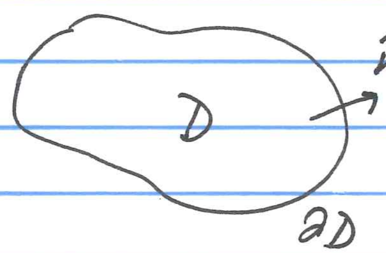
This is a WELL POSED PROBLEM (more - later)

Application to diffusion processes: heat conduction, diffusion of charged particles (e.g. ions, electrons), spread of a contaminant, Markov process (random walk).

How do we arrive at the diffusion eq^{n}?

We consider heat conduction: balance of energy in an arbitrary but fixed control volume.

Take some region D (consider 3D space)



$\hat{n}$: outward unit normal (points outwards)

Denote by Q the thermal energy (heat) in a fixed control volume D

(I) $$Q = \int_D \underbrace{c\rho}\, \theta \, dV \qquad (1.1)$$

thermal energy per unit volume, integrated over volume

c: specific heat

units: energy / (mass . degree of temperature)
$\qquad\qquad\qquad\qquad g \qquad\qquad C^\circ$

ρ: mass density
units: mass / volume

θ: temperature
units: degrees of temperature

$c\rho$: energy per unit volume per degree

Q changes due to:

(a) production of thermal energy inside D (*)

(Q is a function of t alone, V is fixed

(b) heat flux across the boundary ∂D.

Rate of change of Q is due to sum of these two effects:

(II) $$\frac{dQ}{dt} = \int_D P\,dV - \int_{\partial D} \vec{j}\cdot\hat{n}\,dS \qquad (1.2)$$

rate of change of Q — production of thermal energy — heat flux across the boundary

Heat flux: amount of heat transferred per unit area per unit time from or to a surface

P: energy per unit volume produced inside D

$\vec{j}$: heat flux (energy per unit area per time)

$J / m^2 / sec$

Joule

We want to describe $\vec{j}$: heat flux in terms of temperature under reasonable conditions

Fourier's law of heat conduction states (variation of temperature causes heat flux):

(iii) $$\vec{j} = -k \nabla \theta \qquad (1.3)$$

(heat flows from hot to cold)

temperature gradient is in reverse direction

$\vec{j}$: heat flux

$\nabla\theta$: temperature gradient

k : constant of proportionality

THERMAL CONDUCTIVITY

usually constant, $k > 0$

Fourier's law is applicable to most isotropic media, ie. no preferred direction of heat flow.

Let us set up conservation of energy law in integral form.

We assume that D is fixed in space, ∂D is smooth etc. [assume temperature θ is smooth in x & t].

Gauss Thm / Divergence Thm :

$$\int_{\partial D} \vec{j}\cdot\hat{n}\, dS = \int_{D} \nabla\cdot \underbrace{\vec{j}}_{=-k\nabla\theta}\, dV$$

(true for any smooth vector field $\vec{j}$ & steady & smooth surface of D)