1/17/2018

1.1 Fundamental Solution in 1D (Cont'd)

Free-space Green's function in 1D

$$(1.1.1)\quad \begin{cases} u_t - u_{xx} = \delta(x)\,\delta(t) & x \in (-\infty, +\infty) \\ & t > 0 \\ u(x,t) \to 0 \quad \text{as } |x| \to \infty & \\ u(x, 0^-) = 0 & \end{cases}$$

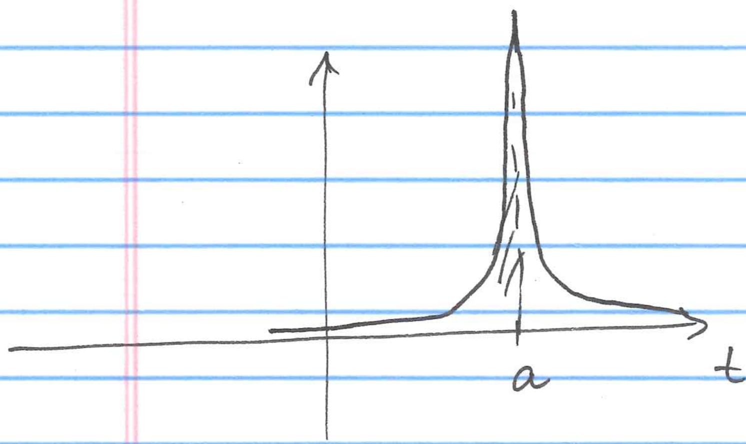
Note

$$\delta(x-a) = \begin{cases} 0, & x \neq a \\ \infty, & x = a \end{cases}$$

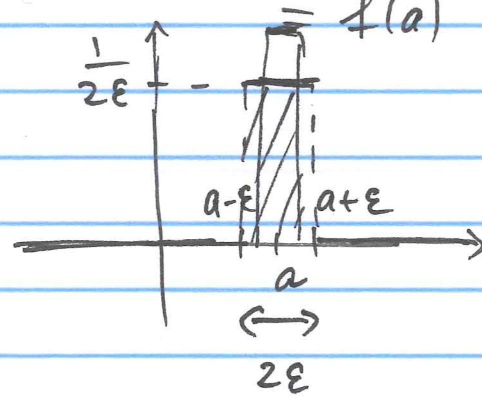
$$\delta_a(x)$$

$$\int_{-\infty}^{\infty} \delta(x-a)\,dx = 1$$

$$\int_{-\infty}^{\infty} \delta(x-a)\,f(x)\,dx = f(a)$$



continuous approximation of $\delta(x-a)$



piecewise constant approximation of $\delta(x-a)$

Here:

(i) δ is the Dirac delta function

(ii) boundaries are far away at ∞: "free space"

Physically: we want to find the temperature in an unbounded 1D medium due to the heat production term, given by $\delta(x)\,\delta(t)$, i.e. the δ function forcing in the differential equation is at $x=0$ and $t=0$.

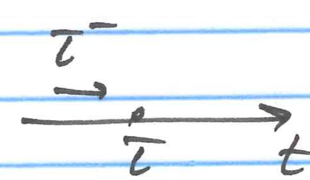
due to concentrated heat source

a translation (change of variables

$x' = x - \xi, \quad t' = t - \tau$) shows that if

$F(x,t)$, solution of (1.1.1.), then the solution of

$$\begin{cases} u_t - u_{xx} = \delta(x-\xi)\,\delta(t-\tau), & \xi, x \in (-\infty, +\infty), \ t > \tau \\ u(x,t) \to 0 \quad \text{as } |x| \to \infty \\ u(x, \tau^-) = 0 \end{cases}$$



is $F(x-\xi, t-\tau)$.

We will use Laplace transform method to find $F(x,t)$.

<u>Note</u> Laplace transform method works only for linear problems similarly to Fourier transforms.

Recall

Let $f(t)$ be some function, $t \geq 0$.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t) e^{-st}\, dt$$: LT Laplace transform of $f(t)$

In our case, $F(x,t)$ is defined on $x \in (-\infty, +\infty)$, $t \geq 0$, so we will apply LT in t-variable.

Let $\mathcal{L} u(x,t) \equiv U(x,s)$

$$\begin{cases} U(x,s) = \displaystyle\int_{0^{(-)}}^\infty u(x,t) e^{-st}\, dt & : \text{Laplace transform of } u(x,t) \\ u(x,t) = \dfrac{1}{2\pi i} \displaystyle\int_\Gamma U(x,s) e^{st}\, ds & : \text{inverse L.T.} \end{cases}$$

s is complex, this is an integral in complex plane

Inverse L.T. $\mathcal{L}^{-1}\,\bar{U}(x,s) = u(x,t)$

This defines a L.T. pair

Space of functions $u(x,t)$ $\underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}}$ space of Laplace transformed functions $U(x,s)$

Γ is the Bromwich contour : a straight line // to the imaginary axis in the complex s-plane which is to the right of all singularities of $U(x,s)$ (& hence of whole integrand)

$$\int_{a-i\infty}^{a+i\infty} \dots \, ds$$

$a \in \mathbb{R}$ is greater than the real part of singularity furthest to the right.

Recall

$$\mathcal{L}\,\frac{\partial u}{\partial t}(x,t) = s\,\mathcal{L}\,u(x,t) - u(x,0^-)$$

$$\mathcal{L}\,\delta(t) = 1$$

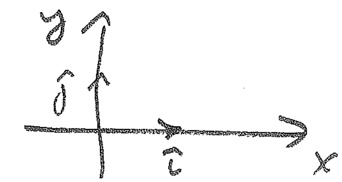
$$\mathcal{L}\,\delta(t) = \int_0^\infty \delta(t)\, e^{-st}\, dt = e^{-st}\Big|_{t=0} = 1$$

$f(t)$ $\qquad F(s) = \mathcal{L}\{f(t)\}$

$$\mathcal{L}\{f'\} = sF(s) - f(0)$$

Note that in our problem $u(x,0^-) = 0$.

14.6 Directional Derivatives and the Gradient Vector



$T = T(x,y)$: temperature

T_x : rate of change of T in positive x-direction, i.e. in the direction of unit vector $\hat{i}$

T_y : rate of change of T in positive y-direction, i.e. in the direction of unit vector $\hat{j}$

Directional derivative : rate of change of a function in the direction given by some unit vector $\vec{u} = \langle a, b \rangle$.

Consider $z = f(x,y)$. Recall

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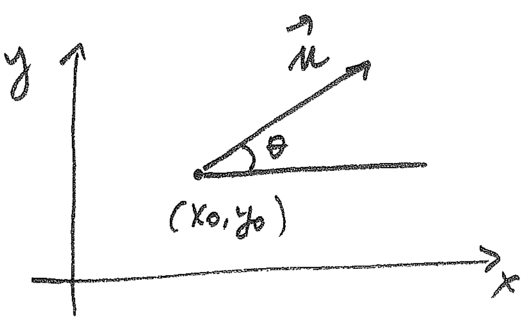
$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

$$f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

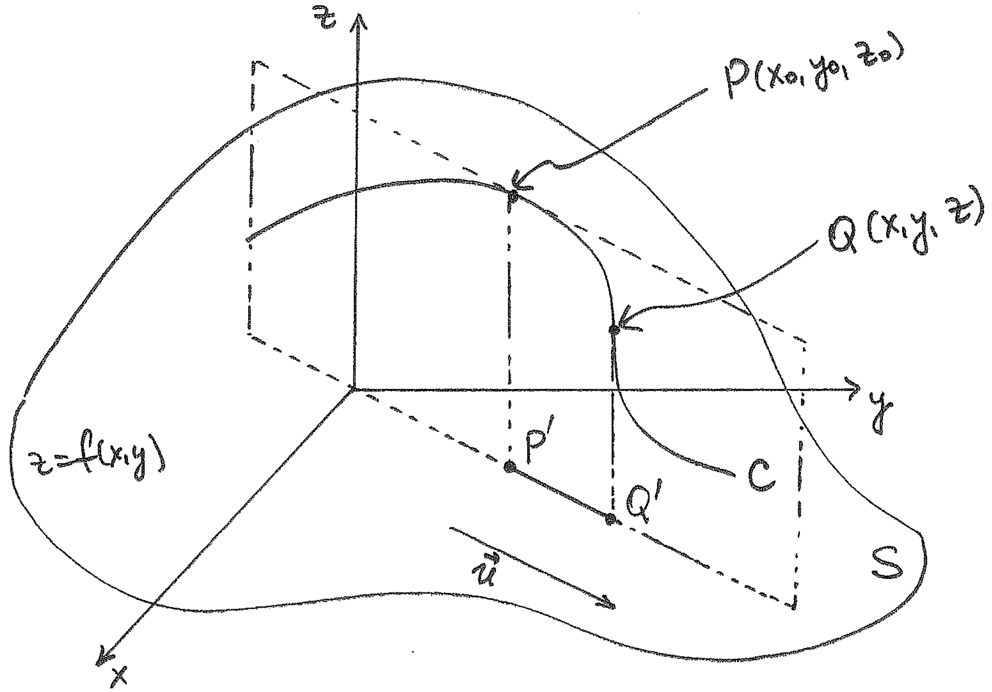


$\vec{u}$: unit vector given by angle θ

$\vec{u} = \langle \cos\theta, \sin\theta \rangle = \langle a, b \rangle$

Graph of $z = f(x,y)$ is surface S.

Directional Derivative (Cont'd)



Graph of $z = f(x,y)$ is surface S.

$P(x_0, y_0, z_0)$: point on S
$z_0 = f(x_0, y_0)$

Take a cross-section of this surface with plane through pt P parallel to $\vec{u}$. The intersection is the curve C.

$Q(x,y,z)$ is another pt on C.

P', Q': projections of P, Q onto xy-plane

Let T: tangent line to curve C

$\overrightarrow{P'Q'} \parallel \vec{u} \Rightarrow \overrightarrow{P'Q'} = h\vec{u}$, h: scalar

$\overrightarrow{P'Q'} = \langle x - x_0, y - y_0 \rangle = h \langle a, b \rangle \Rightarrow x - x_0 = ha, \quad y - y_0 = hb$

or $\boxed{x = x_0 + ha, \quad y = y_0 + hb}$

Now, consider

$$\frac{z-z_0}{h} = \frac{\Delta z}{h} = \frac{f(x,y)-f(x_0,y_0)}{h} = \frac{f(x_0+ha,\, y_0+hb)-f(x_0,y_0)}{h}$$

Take limit as $h \to 0$. This limit is called <u>directional derivative</u> of f in the direction of $\vec{u}$ if this limit exists.

<u>Def</u> The <u>directional derivative</u> of f at (x_0, y_0) in the direction of <u>unit</u> vector $\vec{u} = \langle a, b \rangle$ is

$$D_{\vec{u}} f = \lim_{h \to 0} \frac{f(x_0+ha,\, y_0+hb)-f(x_0,y_0)}{h}$$

if limit exists.

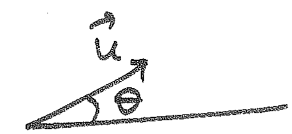
<u>Note</u> if $\vec{u} = \hat{i} = \langle 1, 0 \rangle \Rightarrow D_{\hat{i}} f = f_x$

if $\vec{u} = \hat{j} = \langle 0, 1 \rangle \Rightarrow D_{\hat{j}} f = f_y$

To compute directional derivatives, we use the following result.

Thm If f is a differentiable function of x and y, then f has a directional derivative in the direction of vector $\vec{u} = \langle a, b \rangle$:

$\vec{u} = \langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$



$$\boxed{D_{\vec{u}} f(x,y) = f_x(x,y) \cdot a + f_y(x,y) \cdot b}$$

or

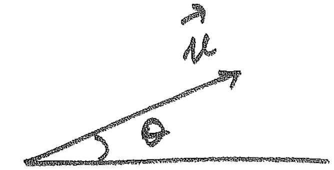
$$\boxed{D_{\vec{u}} f(x,y) = f_x(x,y) \cdot \cos\theta + f_y(x,y) \cdot \sin\theta}$$

if $\vec{u}$ is given by angle θ.

Ex Find $D_{\vec{u}} f(x,y)$ if $f(x,y) = x^2y^3 - y^4$ at $(2,1)$ if $\vec{u}$ is a unit vector given by the angle $\theta = \frac{\pi}{6}$.

$f_x = 2xy^3$, $\quad f_y = 3x^2y^2 - 4y^3$

$f_x(2,1) = 2\cdot 2\cdot 1^3 = 4$, $\quad f_y(2,1) = 3\cdot 2^2\cdot 1^2 - 4\cdot 1^3 = 8$



$\vec{u} = \langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6} \rangle = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$D_{\vec{u}} f(2,1) = f_x(2,1)\cdot \cos\frac{\pi}{6} + f_y(2,1)\cdot \sin\frac{\pi}{6} = 4\cdot\frac{\sqrt{3}}{2} + 8\cdot\frac{1}{2} = \boxed{2\sqrt{3}+4}$

The Gradient Vector

$\vec{u} = \langle a, b \rangle$

Recall

$D_{\vec{u}} f = f_x(x,y)\cdot a + f_y(x,y)\cdot b =$

$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a, b \rangle = \langle f_x, f_y \rangle \cdot \vec{u}$

dot product

Def If f is a function of two variables x and y, the *gradient* of f is the vector ∇f defined by

$$\nabla f = \langle f_x(x,y), f_y(x,y) \rangle = \langle f_x, f_y \rangle = f_x \cdot \vec{i} + f_y \cdot \vec{j}$$

Another notation: $\overrightarrow{\text{grad}}\, f$

Hence,

$$\boxed{D_{\vec{u}} f = \nabla f \cdot \vec{u}}$$

Ex If $f(x,y) = \frac{y^2}{x}$, then

(a) $\nabla f = \langle f_x, f_y \rangle = \langle -\frac{y^2}{x^2}, \frac{2y}{x} \rangle$

(b) $\nabla f \big|_{(1,2)} = \langle -\frac{2^2}{1^2}, \frac{2 \cdot 2}{1} \rangle = \langle -4, 4 \rangle$