1/19/2018

Laplace Transforms

$f(t), \quad t \geq 0$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$: Laplace transform (LT) of $f(t)$

$$\mathcal{L}\{a f(t) + b g(t)\} = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}$$: linearity

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0)$$

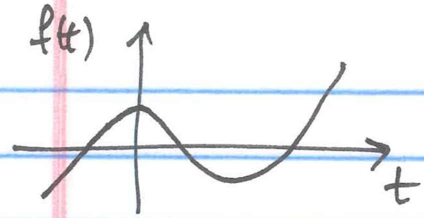
(by integration by parts)

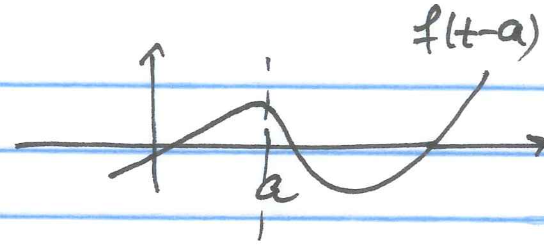
$$\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s f(0) - f'(0)$$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\,\mathcal{L}\{f(t)\} = \frac{F(s)}{s}$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

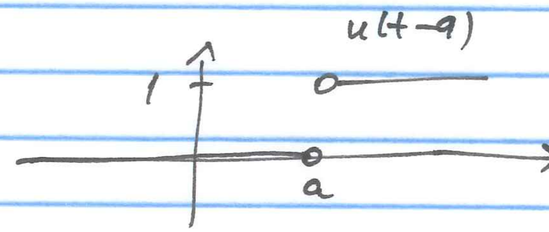
$$\mathcal{L}\{\underbrace{u(t-a)}_{\text{step function}} f(t-a)\} = e^{-as} F(s)$$: shift-chop Thm



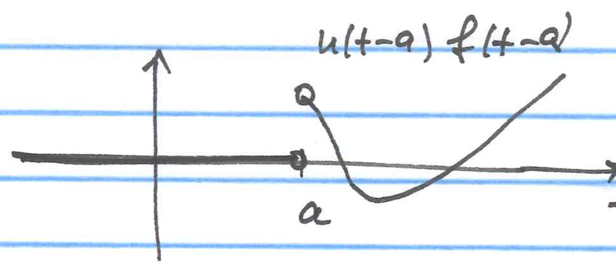


$$u(t-a)=\begin{cases}0, & t<a\\ 1, & t>a\end{cases}$$

unit step function



$u(t-a)\,f(t-a)$



$$(f*g)(t)=\int_0^t f(\tau)\,g(t-\tau)\,d\tau :$$ convolution of f and g

$$f*g=g*f$$

$$\mathcal{L}\{f(t)*g(t)\}=\mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{f\cdot g\}\neq\mathcal{L}\{f\}\cdot\mathcal{L}\{g\}$$

$$\mathcal{L}\{-t\,f(t)\} = F'(s)$$

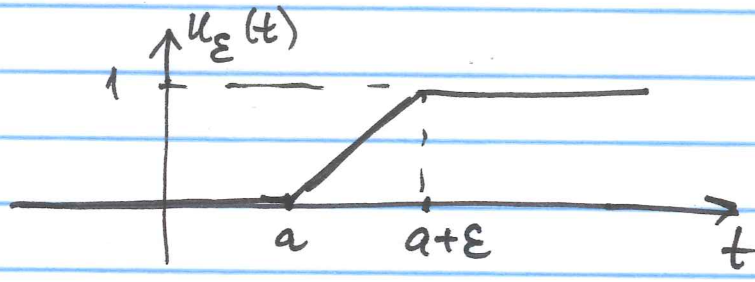
Differentiation of LT

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$$

Integration of LT

Consider



$$\lim_{\varepsilon\to 0} u_\varepsilon(t) = u(t-a)$$

$$u_\varepsilon(t) = \begin{cases} 0, & t < a \\ \frac{1}{\varepsilon}(t-a), & a \le t \le a+\varepsilon \\ 1, & t > a \end{cases}$$

$$\frac{du_\varepsilon}{dt} = \begin{cases} 0, & t < a \\ \frac{1}{\varepsilon}, & a \le t \le a+\varepsilon \\ 0, & t > a \end{cases}$$

$\frac{du_\varepsilon}{dt} \to \infty$ as $\varepsilon \to 0$ at $t = a$

and

$\frac{du_\varepsilon}{dt} = 0,\ t \ne a$

Hence $\boxed{\frac{d}{dt} u(t-a) = \delta(t-a)}$

Back to fundamental solution.

$u = u(x,t)$

$$U(x,s) = \mathcal{L}\{u(x,t)\} = \int_{0^-}^{\infty} u(x,t) e^{-st} dt$$

$$(*) \quad \mathcal{L}\left\{\frac{\partial u}{\partial t}(x,t)\right\} = s\,\mathcal{L}\{u(x,t)\} - \underbrace{u(x,0^-)}_{=0 \text{ given}}$$

We consider the problem

$$\left.\begin{aligned} &u_t - u_{xx} = \delta(x)\delta(t) \\ &u(x,t) \to 0 \quad \text{as } |x| \to \infty \\ &u(x,0^-) = 0 \end{aligned}\right\} \begin{aligned} &\text{solution } u(x,t) \ \exists \\ &\text{and unique and} \\ &u(x,t) \equiv 0, \ t<0 \end{aligned}$$

Apply Laplace transform to both sides of

$$u_t - u_{xx} = \delta(x)\,\delta(t)$$

$$\mathcal{L}\{u_t\} - \mathcal{L}\{u_{xx}\} = \mathcal{L}\{\delta(x)\,\delta(t)\}$$

Use (*)

$$s\,\underbrace{\mathcal{L}\{u(x,t)\}}_{U(x,s)} - \cancel{u(x,0^-)}^{\,0} - U_{xx} = \delta(x)\,\underbrace{\mathcal{L}\{\delta(t)\}}_{=1}$$

$$\int_0^\infty \delta(t)\,e^{-st}\,dt = $$

$$= \int_{-\infty}^{\infty} \delta(t)\,e^{-st}\,dt = 1$$

(**) $\quad sU(x,s) - U_{xx} = \delta(x)$: this is an ODE in x

Since $u(x,t) \to 0$ as $|x| \to \infty$, $U(x,s) \to 0$ as $|x| \to \infty$

when $s > 0$ (i.e. s is on positive real axis to ensure convergence of LT)

Need to solve to find $U(x,s)$. The differential equation (**) implies that $U(x,s)$ is continuous at $x=0$ and U_x has a jump at $x=0$.

Find this jump by $\int_{0^-}^{0^+} \ldots dx$ in (**).

$$-U_x(0^+,s)+U_x(0^-,s)=\underbrace{\int_{0^-}^{0^+}\delta(x)\,dx}_{=1}$$

i.e. $$\Big[U_x(x,s)\Big]_{x=0^-}^{x=0^+} = -1 = \frac{1}{a_2(0)}$$

$a_2(x)$: coefficient of U_{xx}

$[\,..\,]$ denotes a jump $=$ value at 0^+ $-$ value at 0^-

This is a standard notation for quantities with a finite jump at a point.

For $x\neq 0$ $\delta(x)=0$

and $$sU(x,s)-U_{xx}=0$$

or $$U_{xx}-sU=0$$ ODE in x

$s>0$: parameter

has two linearly independent solutions

$$e^{\pm x\sqrt{s}}$$

$$x'' - kx = 0$$

$$r^2 - k = 0$$

$$r = \pm\sqrt{k}$$

$$x = e^{rt}$$

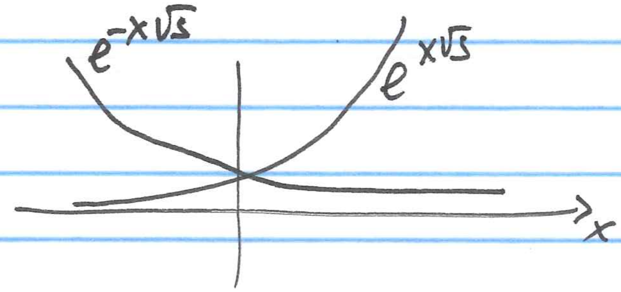
$$x = e^{\pm\sqrt{k}\,t}$$

$\Rightarrow$ General solution is

$$U(x,s) = C_1 e^{x\sqrt{s}} + C_2 e^{-x\sqrt{s}}$$

linear combination of $e^{\pm x\sqrt{s}}$ for $x \neq 0$

BC: $U(x,s) \to 0$ as $|x| \to \infty$ for $s > 0$



$\Rightarrow C_1 = 0$ for $x > 0$ and $C_2 = 0$ for $x < 0$

$$U(x,s) = \begin{cases} C_1 e^{x\sqrt{s}}, & x < 0 \\ C_2 e^{-x\sqrt{s}}, & x > 0 \end{cases}$$

$U(x,s)$ is continuous at $x = 0 \Rightarrow C_1 = C_2 \equiv A$

$$U(x,s) = \begin{cases} A e^{x\sqrt{s}}, & x<0 \\ A e^{-x\sqrt{s}}, & x>0 \end{cases} = A e^{-|x|\sqrt{s}}$$

What about the jump condition?

$$\frac{\partial U(x,s)}{\partial x} = \begin{cases} A\sqrt{s}\, e^{x\sqrt{s}}, & x<0 \\ -A\sqrt{s}\, e^{-x\sqrt{s}}, & x>0 \end{cases}$$

$$\left[\frac{\partial U(x,s)}{\partial x}\right]_{x=0^-}^{x=0^+} = -A\sqrt{s}\, e^{-x\sqrt{s}}\Big|_{x=0^+} -$$

$$-A\sqrt{s}\, e^{x\sqrt{s}}\Big|_{x=0^-} = -2A\sqrt{s} = -1$$

$$\Rightarrow A = \frac{1}{2\sqrt{s}}$$

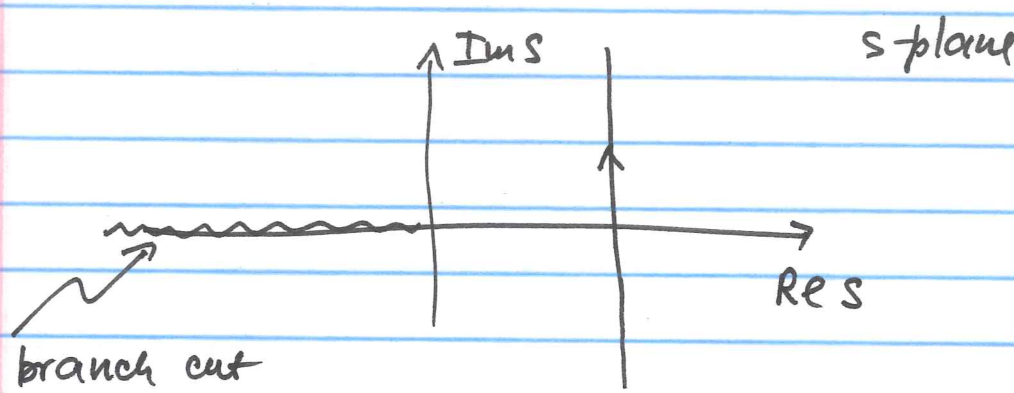
So,

$$U(x,s) = \frac{e^{-|x|\sqrt{s}}}{2\sqrt{s}}$$

This is L.T. of the fundamental solution / Green's f$^{\underline{n}}$

So, we need to perform inversion to get $F(x,t)$. This is NON-TRIVIAL. See pg. 596 of Kevorkian.

$$u(x,t) = \frac{1}{2\pi i} \int_{\Gamma} \bar{U}(x,s)\, e^{st}\, ds$$



Close contour to the right of all singularities