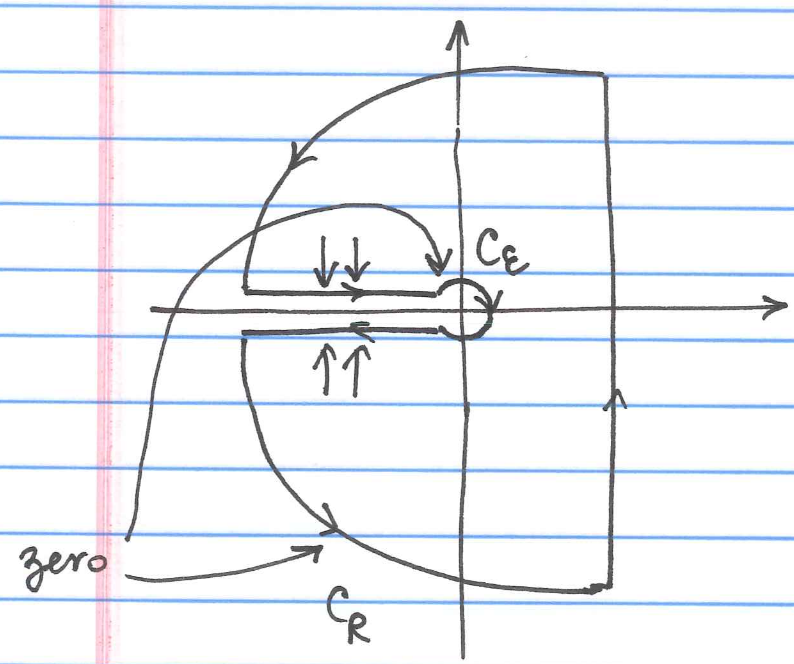
1/22/2018

$$U(x,s) = \frac{e^{-|x|\sqrt{s}}}{2\sqrt{s}}$$

$$u(x,t) = \frac{1}{2\pi i} \int_{\Gamma} U(x,s) e^{st} ds$$

Γ: Bromwich contour



No singularities inside Γ

$\Rightarrow \oint \ldots = 0$

Integral of whole contour $\Rightarrow 0$ (since no residues inside)

$\uparrow\uparrow$ non-zero contribution

It can be shown (see textbook pg. 596) that

$$\mathcal{L}^{-1}\{\bar{U}(x,s)\} = \frac{H(t)\, e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t}}$$

so, free space Green's function is

$$\boxed{F(x,t) = \frac{H(t)\, e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t}}}$$

Here

$$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Heaviside function or unit step function

Note that $F(x,t)$ has a horrible singularity at $x=0$, $t=0$

We have a pair

$$\frac{e^{-|x|\sqrt{s}}}{2\sqrt{s}} \longleftrightarrow H(t)\, \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t}}$$

In general, we have a source at $x=\xi$ and $t=\tau$ so we need a shifted response at $x=\xi$, $t=\tau$. What we need is

$$\boxed{F(x-\xi, t-\tau) = H(t-\tau)\, \frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}}}{2\sqrt{\pi(t-\tau)}}}$$

Note

(1) "Infinite signal speed"

$F(x,t)$ is the temperature due to a point source of thermal energy at $x=0, t=0$.
But $\forall t>0$ (no matter how small) $F(x,t)>0 \ \forall x$!

$\Rightarrow$ an ∞ rod instantly "feels" the effect of a heat source at $x=0$.

We might expect finite diffusion time but in fact this is NOT the case.

(2) Consider fixed x and look at what happens as t gets smaller $\to 0$, i.e. backwards in time.

(1.1.1) $$u_t - u_{xx} = \delta(x)\,\delta(t)$$

$$\int_{0^-}^{0^+} (1.1.1)\, dt$$

$$\int_{0^-}^{0^+} \frac{\partial u}{\partial t}(x,t)\, dt = u(x,0^+) - \cancel{u(x,0^-)}$$

0 given

$$\int_{0^-}^{0^+} \frac{\partial u}{\partial t}(x,t)\, dt \overset{(1.1.1)}{=} \cancel{\int_{0^-}^{0^+} \underbrace{\frac{\partial^2 u}{\partial x^2}(x,t)}_{\text{cts}}\, dt}^{\to 0} + \int_{0^-}^{0^+} \delta(x)\delta(t)\, dt =$$

$$= \delta(x) \underbrace{\int_{0^-}^{0^+} \delta(t)\, dt}_{=1} = \delta(x)$$

This suggests that

$$u(x,0^+) = \delta(x)$$

or $$F(x,0^+) = \delta(x)$$

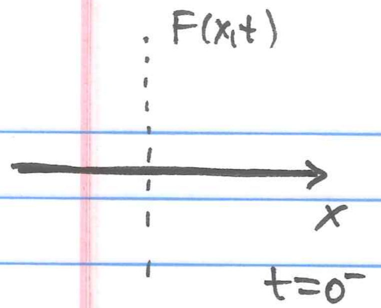
ie.

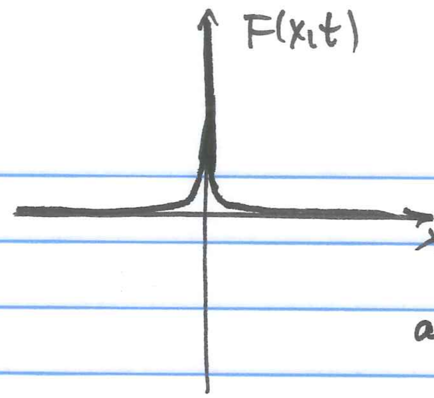
$$\delta(x) = \lim_{t\to 0^+} \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t}}$$

("=" is in the distribution sense only, limit on RHS does not exist in the classical sense)

For $\phi \in \mathcal{D}$ - some domain
test functions

$$\lim_{t\to 0^+} \int_{\mathcal{D}} \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t}} \phi(x)\, dx$$

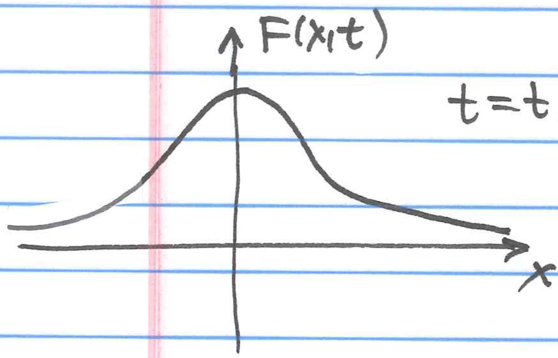


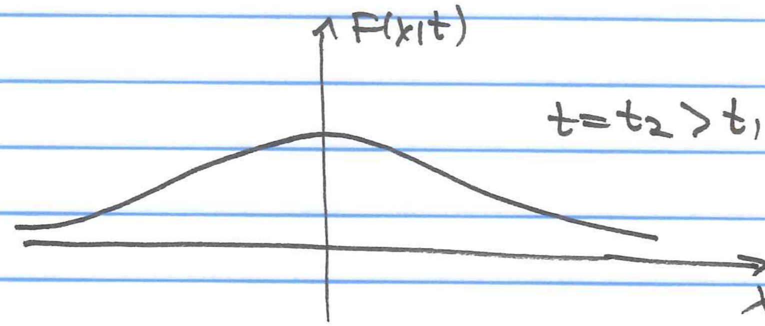


δ function at $x=0$, $t=0$

at $t=0^+$

identically zero for $t<0$





$F(x,t)$ is even in x

decays exp away from $x=0$

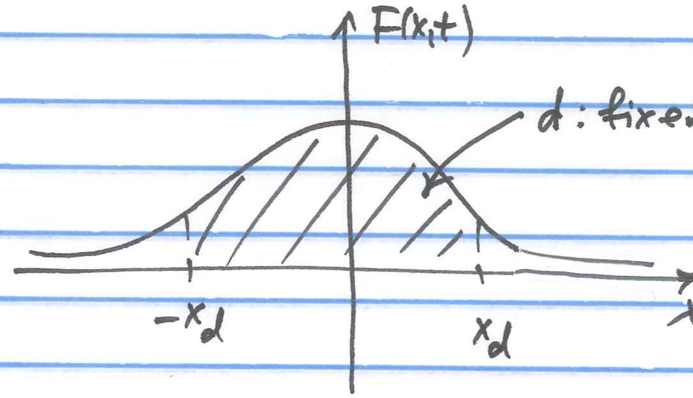
(3) Diffusion of u in space (i.e. in x) is proportional to $\sqrt{t}$.

We consider 1D domain w/ no boundaries.

Let $$d = \int_{-x_d}^{x_d} F(x,t)\,dx$$: given fraction of total heat where $\alpha \in (0,1)$ is a given (fixed) fraction of the total energy (thermal energy if we are thinking about heat)

$$\int_{-\infty}^{\infty} F(x,t)\,dx = 1$$

(no heat loss, so we expect $\int_{-\infty}^{\infty} F$ to be 1)



Note as $t \nearrow$, $x_\alpha \nearrow$ as well (to maintain the same area under $F(x,t)$).

The idea is to find $x_\alpha(t)$. The energy between $-x_\alpha(t)$ and $x_\alpha(t)$ is a function of the total energy.

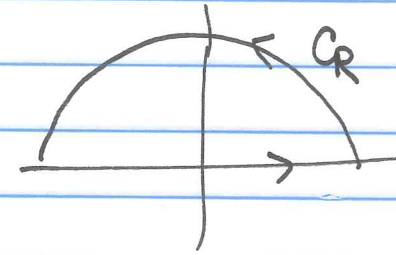
$$\alpha = \int_{-x_\alpha}^{x_\alpha} \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t}}\,dx = 2\int_0^{x_\alpha} \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t}}\,dx = \left| \begin{array}{l} p = \frac{x}{2\sqrt{t}} \\ dp = \frac{dx}{2\sqrt{t}} \end{array} \right|$$

$F(x,t)$ is an even f^n

$$= 2\int_0^{\frac{x_\alpha}{2\sqrt{t}}} \frac{e^{-p^2}}{2\sqrt{\pi t}}\,2\sqrt{t}\,dp = \frac{2}{\sqrt{\pi}}\int_0^{\frac{x_\alpha}{2\sqrt{t}}} e^{-p^2}\,dp =$$

$$= \operatorname{erf}\left(\frac{x_\alpha}{2\sqrt{t}}\right) :$$ error function

Note that $\displaystyle\int_0^\infty e^{-p^2}\,dp = \frac{\sqrt{\pi}}{2}$



Jordan's lemma

Hence,

$$d = \operatorname{erf}\left(\frac{x_d}{2\sqrt{t}}\right)$$

$d = \text{const} \Rightarrow \dfrac{x_d}{2\sqrt{t}} = \text{const}$

so as $t \nearrow$, $x_d \sim \sqrt{t}$

proportional

we say that heat due to a source at $x=0$, $t=0$ diffuses according to $|x| \sim \sqrt{t}$, i.e. if d is fixed, then $x_d \propto \sqrt{t}$

proportional