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Finding $F(x,t)$ by similarity method

This technique is not confined to linear problems, but similarity method works well when geometry and boundary conditions are sufficiently simple. Solutions obtained by this method are called <u>similarity solutions</u>.

We will consider 1D problem for heat equation w/ homogeneous data on infinite domain.

Let $F(x,t)$ be the Green's function in free space of

$$u_t - u_{xx} = \delta(x)\delta(t)$$

$$u(x,0^-) = 0$$

$$u(x,t) \to 0 \quad \text{as} \quad |x| \to \infty$$

Look for invariance properties of this problem under rescaling (applied to PDE, BCs and ICs), i.e.

$$G(x,t) = \alpha F(\beta x, \gamma t)$$

where α, β, γ are positive constants, and we require $G(x,t)$ to be a solution of the same problem. If so, we can determine F using uniqueness of the solution.

Scaling $\rightarrow$ magnification / shrinking

Let

$$\bar{x} = \beta x, \qquad \bar{t} = \gamma t$$

Then

$$G(x,t) = \alpha F(\bar{x}, \bar{t})$$

$$G_t = \frac{\partial G}{\partial t} = \overbrace{\frac{\partial G}{\partial \bar{t}}}^{=\alpha F(\bar{x},\bar{t})} \cdot \underbrace{\frac{d\bar{t}}{dt}}_{=\gamma} = \alpha \cdot F_{\bar{t}} \cdot \gamma = \alpha\gamma F_{\bar{t}}$$

$$G_x = \frac{\partial G}{\partial x} = \overbrace{\frac{\partial G}{\partial \bar{x}}}^{=\alpha F(\bar{x},\bar{t})} \cdot \underbrace{\frac{d\bar{x}}{dx}}_{\beta} = \alpha F_{\bar{x}}\, \beta = \alpha\beta F_{\bar{x}}$$

$$G_{xx} = \alpha \beta^2 F_{\bar{x}\bar{x}}$$

$G(x,t)$ is a solution of the same problem

$$G_t - G_{xx} = \delta(x)\,\delta(t)$$

$$G(x, 0^-) = 0$$

$$G(x,t) \to 0 \quad \text{as } |x| \to \infty$$

Rewrite this problem in terms of $F, \bar{x}, \bar{t}$.

Recall

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

$$\bar{x} = \beta x \quad \Rightarrow \quad x = \frac{\bar{x}}{\beta} \qquad \delta(x) = \delta\left(\frac{\bar{x}}{\beta}\right) = \beta\delta(\bar{x})$$

$$\bar{t} = \gamma t \quad \Rightarrow \quad t = \frac{\bar{t}}{\gamma} \qquad \delta(t) = \delta\left(\frac{\bar{t}}{\gamma}\right) = \gamma\delta(\bar{t})$$

$G_t - G_{xx} = \delta(x)\,\delta(t)$ becomes

$$(*) \quad \alpha\gamma F_{\bar{t}}(\bar{x},\bar{t}) - \alpha\beta^2 F_{\bar{x}\bar{x}}(\bar{x},\bar{t}) = \beta\gamma\,\delta(\bar{x})\delta(\bar{t}) \;\Big|\; \frac{1}{\alpha\gamma}$$

and also

BC $\quad \alpha F(\bar{x},\bar{t}) \to 0$ as $|\bar{x}| \to \infty$

IC $\quad \alpha F(\bar{x},0^-) = 0$

multiply $(*)$ by $\frac{1}{\alpha\gamma}$

$$(**) \quad \therefore\ F_{\bar{t}}(\bar{x},\bar{t}) - \frac{\alpha\beta^2}{\alpha\gamma} F_{\bar{x}\bar{x}}(\bar{x},\bar{t}) = \frac{\beta\gamma}{\alpha\gamma}\delta(\bar{x})\delta(\bar{t})$$

$$F_{\bar{t}}(\bar{x},\bar{t}) - \frac{\beta^2}{\gamma} F_{\bar{x}\bar{x}}(\bar{x},\bar{t}) = \frac{\beta}{\alpha}\delta(\bar{x})\delta(\bar{t})$$

$F(\bar{x},\bar{t})$ is also a solution of the problem but in terms of $\bar{x}, \bar{t}$, i.e. we need

$$F_{\bar{t}} - F_{\bar{x}\bar{x}} = \delta(\bar{x})\,\delta(\bar{t}), \qquad F(\bar{x},\bar{t}) \to 0 \text{ as } |\bar{x}| \to \infty$$
$$F(\bar{x}, 0^-) = 0$$

From eq^{n} $(**)$ we see that we must have

$$\frac{\beta^2}{\gamma} = 1 \quad \text{and} \quad \frac{\beta}{\alpha} = 1$$

$$\Rightarrow \boxed{\beta = \alpha \quad \text{and} \quad \gamma = \beta^2 = \alpha^2}$$

i.e. there is a relation between magnification constants α, β, γ.

Hence, we can write

$$G(x,t) = \alpha F(\alpha x, \alpha^2 t)$$

i.e. if $F(x,t)$ is a solution, then $\alpha F(\alpha x, \alpha^2 t)$ is also a solution. Since solution is unique, $G \equiv F$ and we say that F has similarity property

$$\alpha F(\alpha x, \alpha^2 t) = F(x,t)$$

This gives information about $F(x,t)$:
if we multiply x by α, t by α^2 and then multiply $F(\alpha x, \alpha^2 t)$ by α, we get $F(x,t)$.

$$\alpha x \qquad \frac{(\alpha x)^2}{\alpha^2 t} = \frac{\cancel{\alpha^2} x^2}{\cancel{\alpha^2} t} = \frac{x^2}{t} \quad \text{or} \quad \frac{x}{\sqrt{t}}$$

ie. these ratios do not change under scaling

similarly,

$$\frac{\alpha F(\alpha x, \alpha^2 t)}{\alpha x} = \frac{F}{x} \quad \text{or} \quad \frac{\alpha F}{\sqrt{\alpha^2 t}} = \frac{F}{\sqrt{t}}$$

these ratios also do not change under scaling.

So, this means that whenever x appears in F, it must be divided by $\sqrt{t}$ to have $\frac{x}{\sqrt{t}}$ and function must be premultiplied by either a factor of $\frac{1}{x}$ or $\frac{1}{\sqrt{t}}$, ie.

$$F(x,t) = \frac{1}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right) \quad \text{or}$$

$$F(x,t) = \frac{1}{\sqrt{t}} g\left(\frac{x^2}{t}\right) \quad \text{or} \quad F(x,t) = \frac{1}{x} h\left(\frac{x}{\sqrt{t}}\right)$$