1/26/2018

We showed last time that under scaling

$$F(x,t) = \alpha F(\alpha x, \alpha^2 t)$$

the ratios $\frac{(\alpha x)^2}{\alpha^2 t} = \frac{x^2}{t}$ or $\frac{\alpha x}{\sqrt{\alpha^2 t}} = \frac{x}{\sqrt{t}}$

and $\frac{\alpha F}{\alpha x} = \frac{F}{x}$ or $\frac{\alpha F}{\sqrt{\alpha^2 t}} = \frac{F}{\sqrt{t}}$

stay the same, hence $F(x,t)$ can be written as

$$F(x,t) = \frac{1}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right) \quad \text{or} \quad F(x,t) = \frac{1}{\sqrt{t}} g\left(\frac{x^2}{t}\right)$$

or $F(x,t) = \frac{1}{x} h\left(\frac{x}{\sqrt{t}}\right)$

The non-uniqueness is not a problem, and we can choose any of these functions f, g or h. For example, let

$$F(x,t) = \frac{1}{\sqrt{t}} f(\zeta), \quad \zeta = \frac{x}{\sqrt{t}}$$

$F(x,t)$ satisfies

$$F_t - F_{xx} = \delta(x)\,\delta(t)$$

(1)

If $t \neq 0 \Rightarrow F_t - F_{xx} = 0$ homog. PDE

$$F_t = \frac{f'(\xi)\frac{\partial \xi}{\partial t}\cdot\sqrt{t} - f(\xi)\cdot\frac{1}{2\sqrt{t}}}{t} =$$

$' = \frac{d}{d\xi}$

$$\xi = \frac{x}{\sqrt{t}} \qquad \frac{\partial \xi}{\partial t} = -\frac{1}{2}\frac{x}{t^{3/2}} \qquad \frac{\partial \xi}{\partial x} = \frac{1}{\sqrt{t}}$$

$$= \frac{f'(\xi)\cdot\left(-\frac{1}{2}\right)\frac{x}{t\cdot\cancel{\sqrt{t}}}\cancel{\sqrt{t}} - f(\xi)\frac{1}{2\sqrt{t}}}{t} =$$

$$= \frac{1}{t}\left[-\frac{x f'}{2t} - \frac{f}{2t^{1/2}}\right] = -\frac{x f'}{2t^2} - \frac{f}{2t^{3/2}}$$

$$F_x = \frac{1}{\sqrt{t}} f'(\xi)\cdot\underbrace{\frac{\partial \xi}{\partial x}}_{\sqrt{t}} = \frac{1}{t} f'(\xi)$$

$$F_{xx} = \frac{1}{t^{3/2}} f''(\xi)$$

Substitute F_t, F_{xx} into (1).

$$(-t^{3/2})\cdot\Big| \quad \underbrace{-\frac{x f'}{2t^2} - \frac{f}{2t^{3/2}}}_{F_t} - \underbrace{\frac{1}{t^{3/2}} f''(\xi)}_{F_{xx}} = 0 \quad \text{if } t \neq 0$$

$$f'' + \underbrace{\frac{x}{2\sqrt{t}}}_{=\zeta} f' + \frac{f}{2} = 0 \qquad t \neq 0 \qquad \zeta = \frac{x}{\sqrt{t}}$$

(2) $$f'' + \frac{\zeta}{2} f' + \frac{f}{2} = 0$$ ODE for f 2nd order

Note $\zeta f' + f = (\zeta f)'$

$\Rightarrow$ we can write (2) as

$$f'' + \frac{1}{2}(\zeta f)' = 0$$

Integrate once

$$f' + \underbrace{\frac{1}{2}\zeta}_{P(\zeta)} f = \overbrace{A}^{Q(\zeta)}$$

A const of integration

$$\rho(\zeta) = e^{\int \frac{1}{2}\zeta \, d\zeta} = e^{\frac{\zeta^2}{4}}$$

$$\rho \cdot f = \int \rho A \, d\zeta + B$$

B const

$$e^{\frac{\zeta^2}{4}} \cdot f = \int e^{\frac{\zeta^2}{4}} \cdot A \, d\zeta + B \quad \Big| \; e^{-\frac{\zeta^2}{4}}$$

$$\boxed{f(\zeta) = A e^{-\frac{\zeta^2}{4}} \int^{\zeta} e^{\frac{s^2}{4}} \, ds + B e^{-\frac{\zeta^2}{4}}}$$

Aside

Method of integrating factor for 1st order linear ODEs:

$$y' + P(x) y = Q(x)$$

$$\rho(x) = e^{\int P(x)\,dx} : \text{integrating factor}$$

$$\rho y = \int \rho Q \, dx + C$$

Solve for y

$$F = \frac{f(\zeta)}{\sqrt{t}}$$

We can find constants A and B by considering the total heat

$$H(t) = \int_{-\infty}^{\infty} F(x,t)\,dx \;\boxed{=}$$

integral of temperature over the entire domain

total heat

Let

$$f_1(\zeta) = e^{-\frac{\zeta^2}{4}} \int^{\zeta} e^{\frac{s^2}{4}}\,ds = \left| \begin{array}{l} \sigma = \frac{s^2}{4} \\ s = 2\sqrt{\sigma} \\ ds = \frac{d\sigma}{\sqrt{\sigma}} \end{array} \right| =$$

$$= e^{-\frac{\zeta^2}{4}} \int^{\frac{\zeta^2}{4}} e^{\sigma}\,\sigma^{-1/2}\,d\sigma \underset{\substack{\text{by parts} \\ \text{twice}}}{\overset{\text{integration}}{=}} \ldots$$

we can show that

$$(*) \quad f_1(\zeta) = \frac{2}{|\zeta|} + O\left(\frac{1}{\zeta^3}\right) \quad \text{as } |\zeta| \to \infty$$

$$\zeta = \frac{x}{\sqrt{t}} \quad \boxed{=} \underbrace{\frac{A}{\sqrt{t}} \int_{-\infty}^{\infty} f_1\left(\frac{x}{\sqrt{t}}\right) dx}_{\text{I}} + \frac{B}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}}\,dx$$

$$f_1(\xi) = f_1\left(\frac{x}{\sqrt{t}}\right) \overset{(*)}{=} \underbrace{\frac{2\sqrt{t}}{x}}_{\substack{\text{not defined} \\ \text{at } x=0}} \left(1 + O\left(\frac{1}{\xi_0^2}\right)\right)$$

$\Rightarrow \int_{-\infty}^{\infty} f_1(\xi)\, d\xi$ blows up at the origine

since it behaves as $\int \frac{dx}{x}$

$$\Rightarrow A = 0$$

Hence,

$$H(t) = \frac{B}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}}\, dx \quad \Rightarrow \quad 1 = \frac{B}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}}\, dx$$

$$\underbrace{H(t)}_{\substack{\text{will justify this} \\ \text{later}}} = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \qquad \text{Heaviside function}$$

Then we can write

$$1 = \frac{B}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}}\, dx = 2B \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4t}}\, dx =$$

$$= \left| \xi = \frac{x}{2t} \right| = 2B \underbrace{\int_{-\infty}^{\infty} e^{-\xi^2} d\xi}_{\sqrt{\pi}} = 2B\sqrt{\pi}$$

$$\Rightarrow B = \frac{1}{2\sqrt{\pi}}$$

$$\therefore F(x,t) = B e^{-\frac{\eta^2}{4}} \cdot \frac{1}{\sqrt{t}} = \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}}$$

or

$$\boxed{F(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}}$$

Show that $H(t)$ above is the Heaviside function.

Recall that the fundamental solution satisfies

$$u_t - u_{xx} = \delta(x)\delta(t)$$

$$H(t) = \int_{-\infty}^{\infty} F(x,t)\,dx$$

$$F_t - F_{xx} = \delta(x)\delta(t)$$

$$\frac{dH}{dt} = \int_{-\infty}^{\infty} F_t(x,t)\,dx = \int_{-\infty}^{\infty} \left[F_{xx}(x,t) + \delta(x)\delta(t) \right] dx$$

$$= F_x(x,t)\Big|_{-\infty}^{\infty} + \delta(t)\underbrace{\int_{-\infty}^{\infty}\delta(x)\,dx}_{=1} = F_x(\cancel{+\infty},t) - F_x(\cancel{-\infty},t)$$

since
$u \to 0$ as $|x| \to \infty$
u is smooth

$$+\ \delta(t) = \delta(t)$$

$\therefore \quad \dfrac{dH}{dt} = \delta(t) \implies H(t)$ is a unit step function

ie. $H(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$

ie. $H(t) = 1, \quad t>0.$