1/31/2018

# Initial Value Problem on Infinite Domain

Consider

$$(1.2.1)\quad \begin{cases} u_t - u_{xx} = p(x,t) & x\in(-\infty,\infty),\ t>0 \\ u(x,0) = f(x) \end{cases}$$

Equation is nonhomogeneous, IC is nonhomog. as well. Since the IVP is linear, we will use the principle of linear superposition to decompose (1.2.1) into two problems with $p$ and $f$ appearing separately.

Let $u_I$ be the solution of

$$(1.2.2)\quad \begin{cases} u_t - u_{xx} = p(x,t) & x\in(-\infty,\infty),\ t>0 \\ u(x,0^-) = 0 \end{cases}$$

and let $u_{\overline{I}}$ be the solution of

$$(1.2.3)\quad \begin{cases} u_t - u_{xx} = 0 & x\in(-\infty\ \infty),\ t>0 \\ u(x,0^+) = f(x) \end{cases}$$

Then the solution of (1.2.1) is

$$u = u_{I} + u_{II}$$

Solutions to (1.2.2) and (1.2.3) are unique $\Rightarrow$ solution of (1.2.1) is also unique.

Let us show that solution $u_I$ is the superposition integral

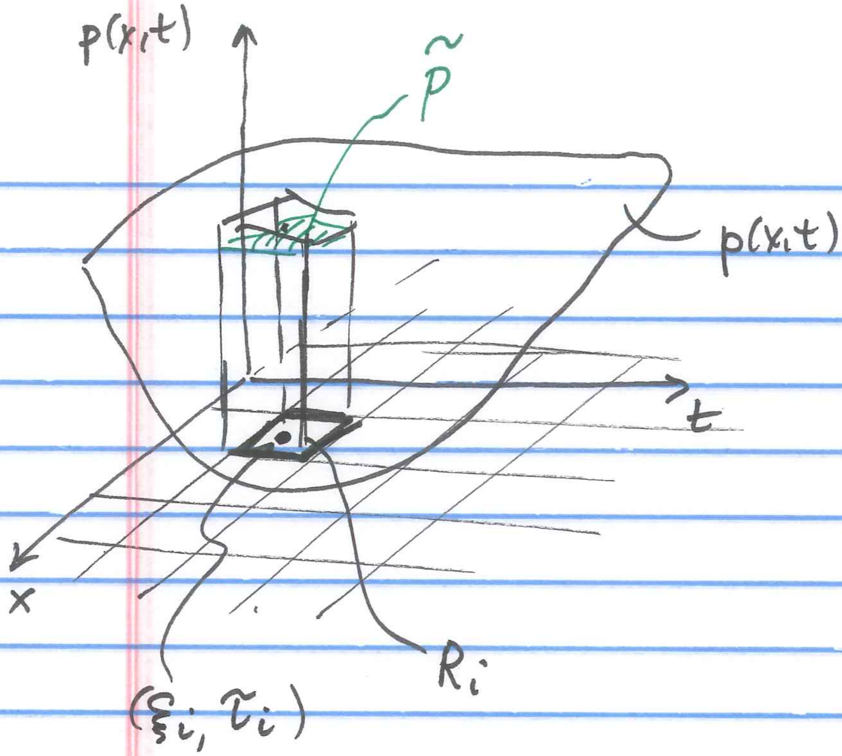
$$u_{I}(x,t) = \int_{0}^{t} \int_{-\infty}^{\infty} p(\xi, \tau)\, F(x-\xi, t-\tau)\, d\xi\, d\tau$$

where $F(x-\xi, t-\tau)$ is the fundamental solution.

Recall that $F(x-\xi, t-\tau)$ is the solution of

$$u_t - u_{xx} = \delta(x-\xi, t-\tau)$$

For this problem, the unit response is at $x=\xi$, $t=\tau$.




Graph of $p(x,t)$ is a surface

Partition $xt$-plane into strips in $x$ and $t$ directions, and approximate function $p(x,t)$ by a piecewise constant function.

Consider some control partition rectangle $R_i$:

$$x \in (\xi, \xi + \Delta\xi), \qquad t \in (\tau, \tau + \Delta\tau)$$

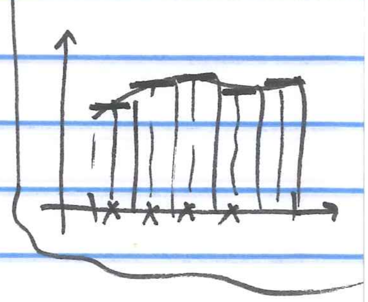
Let $\xi_i, \tau_i$ be some representative points on $R_i$.

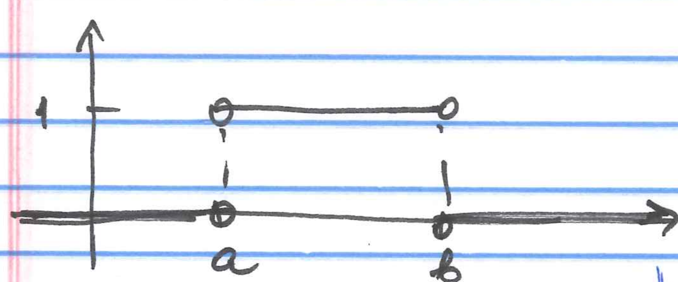
Then

$$p(x,t) \approx p(\xi_i, \tau_i) \equiv \tilde{p}(x,t)$$

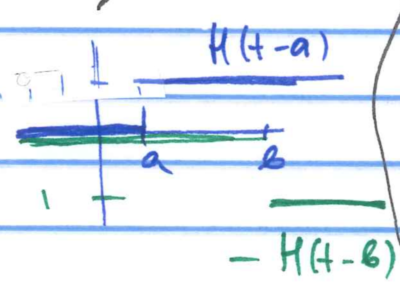
Aside

in 2D



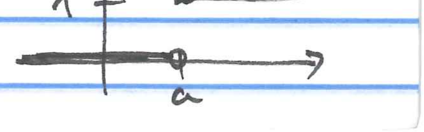



$H(t-a) - H(t-b)$




Recall

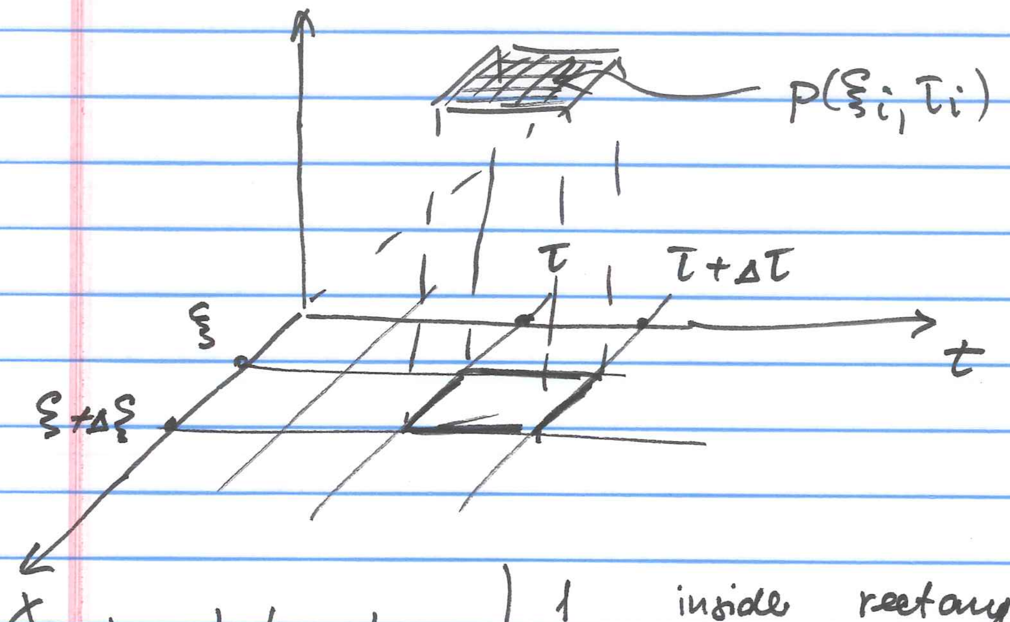
$$H(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a \end{cases}$$



On rectangle $R_i$:

$$\tilde{p}(x,t) = p(\xi_i, \tau_i) \cdot \left\{ H(t-\tau) - H(t-\tau-\Delta\tau) \right\} *$$

$$* \left\{ H(x-\xi) - H(x-\xi-\Delta\xi) \right\} =$$




$$\{\dots\}\{\dots\} = \begin{cases} 1 & \text{inside rectangle } R_i \\ 0 & \text{outside rectange } R_i \end{cases}$$

Recall

$$\frac{dH}{ds} = \delta(s)$$

In the limit as $\Delta\tau \to 0$, $\Delta\xi \to 0$

$$= p(\xi_i, \tau_i) \cdot \delta(t-\tau)\, \delta(x-\xi)\, \Delta\tau\, \Delta\xi +$$

$$+ o(\Delta\tau)\, o(\Delta\xi)$$

Note $\quad \dfrac{H(t-\tau) - H(t-\tau-\Delta\tau)}{\Delta\tau} \xrightarrow[\Delta\tau \to 0]{} \dfrac{dH}{d\tau}$

Now, let's consider

$$u_t - u_{xx} = \underbrace{p(\xi_i, \tau_i)\,\Delta\tau\,\Delta\xi}_{=\tilde{p}\ \text{const}}\ \delta(x-\xi)\,\delta(t-\tau)$$

const

The response to this forcing with $\tilde{p}$ is scaled Green's function

$$F(x-\xi, t-\tau) \cdot p(\xi_i, \tau_i)\,\Delta\tau\,\Delta\xi$$

Now, sum responses due to contributions from all infinitesimal rectangles in the partition to find $u_I$

$$u_I(x,t) = u(x,t) = \int_0^t \int_{-\infty}^{\infty} p(\xi,\tau)\,F(x-\xi, t-\tau)\,d\xi\,d\tau$$

Since

$$F(x-\xi, t-\tau) = \frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}}}{2\sqrt{\pi(t-\tau)}}$$

we have

(1.2.4) $$\boxed{u_I(x,t) = \int_0^t \int_{-\infty}^{\infty} p(\xi,\tau) \frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}}}{2\sqrt{\pi(t-\tau)}} d\xi\, d\tau}$$

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x,0^+) = f(x) \end{cases} \quad (1.2.3)$$

$\underline{u_{II}}$ The problem (1.2.3) is equivalent to

(*) $$\begin{cases} u_t - u_{xx} = f(x)\,\delta(t) \\ u(x,0^-) = 0 \end{cases}$$

Indeed,

$$\int_{0^-}^{0^+} u_t\, dt = \Big[u(x,t)\Big]\Big|_{t=0^-}^{t=0^+} = u(x,0^+) - \cancel{u(x,0^-)}^{\text{given } 0}$$

$$= u(x,0^+)$$

$$\int_{0^-}^{0^+} \Big[\underbrace{u_{xx}}_{\substack{\text{cts} \\ \text{no contribution}}} + f(x)\,\delta(t)\Big] dt = f(x) \underbrace{\int_{0^-}^{0^+} \delta(t)\, dt}_{=1} = f(x)$$

$\Rightarrow u(x, 0^+) = f(x)$ ✓

Problem ⊛ is similar to problem (1.2.2)

but with

$$p(x,t) = f(x)\,\delta(t)$$

Hence,

$$u_{\text{II}}(t) = u(x,t) = \int_0^t \int_{-\infty}^{\infty} f(\xi)\,\delta(\tau)\,\frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}}}{2\sqrt{\pi(t-\tau)}}\,d\xi\,d\tau$$

Recall

$$\int g(\xi)\,\delta(x-\xi)\,d\xi = g(x)$$