Math 540 - Partial Differential Equations - Spring 2018

<u>Homework 1</u>

Due: **Friday, February 9, 2018**

1. Let $L = \frac{d^2}{dx^2} - 2\frac{d}{dx} + 2$, with the boundary conditions $u(0) = 0$, $\frac{du}{dx}(\pi/4) = 0$. Show that the completely homogeneous system has only the trivial solution. Construct the Green's function $G(x, \xi)$.

2. Construct the Green's function for $L = (\frac{d^2}{dx^2}) + k^2$, where $k^2 > 0$, with the boundary conditions $u(0) = 0$, $u(1) = 0$. What are the values of k^2 for which the construction is impossible?

3. Prove the result on the solution of the initial value problem

$$L(u) = f(t), \quad t \geq 0$$

$$u(0) = 0, \; u'(0) = 0$$

in the form of the superposition integral (A.1.67):

$$u(t) = \int_0^t G(t, \tau) f(\tau) d\tau$$

given on page 588.

4. Derive Laplace transform properties:

$$\mathcal{L}(g') = s\mathcal{L}(g) - g(0^+), \quad \mathcal{L}(g'') = s^2\mathcal{L}(g) - sg(0^+) - g'(0^+)$$

where Laplace transform $\mathcal{L}$ of a function $g(t)$ is defined as

$$\mathcal{L}(g) \equiv G(s) = \int_0^\infty g(t)\, \mathrm{e}^{-st}\, dt.$$

<u>Hint</u>: use the definition of Laplace transform and integration by parts.

5. (1.2.1) Consider the diffusion equation with variable coefficients

$$2xu_t - u_{xx} = 0, \quad 0 \leq x < \infty, \; t \geq 0$$

with boundary conditions

$$u(0, t) = C_1 = \text{constant} \quad \text{if } t > 0,$$

$$u(\infty, t) = C_2 = \text{constant} \quad \text{if } t > 0,$$

and initial condition

$$u(x, 0) = C_3 = \text{constant}.$$

   (a) What is the most general choice for the constants C_1, C_2, and C_3 for which the solution of the above initial- and boundary-value problem can be obtained in similarity form?

   (a) For the choice of constants obtained in part (a), calculate the solution and evaluate all integration constants explicitly.