4th order Runge-Kutta method

EXAMPLE

Solve approximately

$$\frac{dy}{dx} = x + \sqrt{y}, \quad y(1) = 2$$

and find $y(1.4)$ in 2 steps using the 4th order Runge-Kutta method.

$$y_{n+1} = y_n + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$$

$$k_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2)$$

$$k_4 = f(x_n + h, y_n + hk_3)$$

At x_0, y_0: $x_0 = 1$, $y_0 = 2$. Also, $f(x, y) = x + \sqrt{y}$.

$$k_1 = f(x_0, y_0) = x_0 + \sqrt{y_0} = 1 + \sqrt{2} = 2.4142$$

$$k_2 = f(x_0 + 0.1, y_0 + 0.1k_1) = 1.1 + \sqrt{2 + 0.1(2.4142)} = 2.5971$$

$$k_3 = f(x_0 + 0.1, y_0 + 0.1k_2) = 1.1 + \sqrt{2 + 0.1(2.5971)} = 2.6032$$

$$k_4 = f(x_0 + 0.2, y_0 + 0.2k_3) = 1.2 + \sqrt{2 + 0.2(2.6032)} = 2.7877$$

$$y_1 = y_0 + \frac{0.2}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 2 + \frac{0.2}{6}(2.4142 + 2(2.5971) + 2(2.6032) + 2.7877) = 2.5201$$

$$x_1 = x_0 + h = 1 + 0.2 = 1.2$$

Next step: use x_1 and y_1 instead of x_0 and y_0.

$$k_1 = f(x_1, y_1) = x_1 + \sqrt{y_1} = 1.2 + \sqrt{2.5201} = 2.7875$$

$$k_2 = f(x_1 + 0.1, y_1 + 0.1k_1) = 1.3 + \sqrt{2.5201 + 0.1(2.7875)} = 2.9730$$

$$k_3 = f(x_1 + 0.1, y_1 + 0.1k_2) = 1.3 + \sqrt{2.5201 + 0.1(2.9730)} = 2.9785$$

$$k_4 = f(x_1 + 0.2, y_1 + 0.2k_3) = 1.4 + \sqrt{2.5201 + 0.2(2.9785)} = 3.1652$$

$$y_2 = y_1 + \frac{0.2}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 2 + \frac{0.2}{6}(2.7875 + 2(2.9730) + 2(2.9785) + 3.1652) = 3.1153$$

$$x_2 = x_1 + h = 1.2 + 0.2 = 1.4$$

Hence,

$$y(1.4) \approx y_2 = 3.1153$$

Here is a simple matlab code (without subroutines):

```
clear all

x0=1; y0=2; h=0.2;

x=x0; y=y0;

k1=x+sqrt(y)
k2=x+h/2+sqrt(y+h/2*k1)
k3=x+h/2+sqrt(y+h/2*k2)
k4=x+h+sqrt(y+h*k3)

ynext=y+h/6*(k1+2*k2+2*k3+k4)

x=x+h; y=ynext;

k1=x+sqrt(y)
k2=x+h/2+sqrt(y+h/2*k1)
k3=x+h/2+sqrt(y+h/2*k2)
k4=x+h+sqrt(y+h*k3)

ynext=y+h/6*(k1+2*k2+2*k3+k4)
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