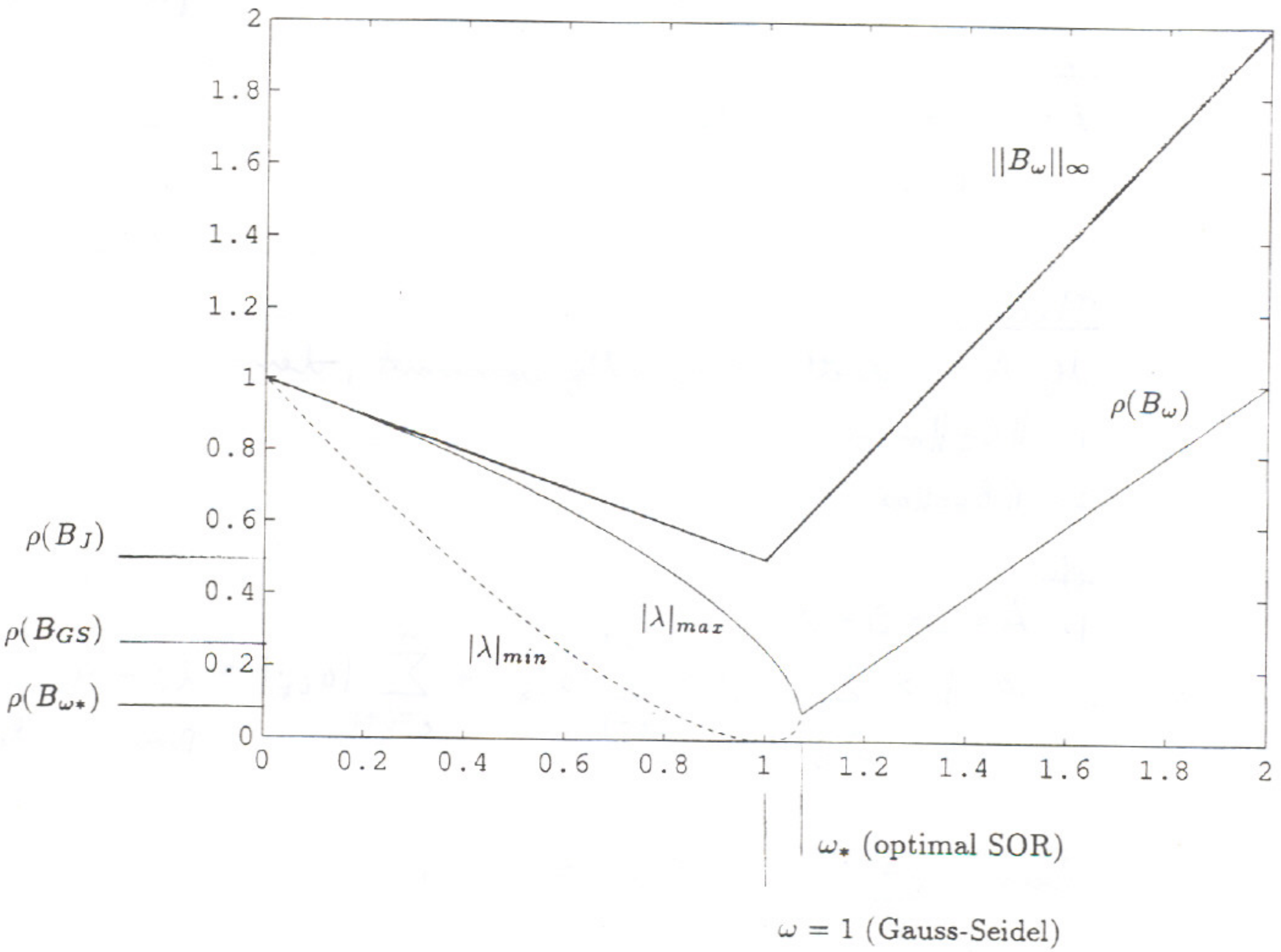
Analysis of the SOR iteration matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \implies B_\omega = \begin{pmatrix} 1-\omega & \omega/2 \\ \omega(1-\omega)/2 & \omega^2/4+1-\omega \end{pmatrix}$$



1. The optimal SOR parameter is $\omega_* = 4/(2+\sqrt{3}\,) \sim 1.0718$.

2. B_ω has 2 eigenvalues.

   For $\omega < \omega_*$, the eigenvalues are real and distinct, $|\lambda|_{min} < |\lambda|_{max}$.

   For $\omega = \omega_*$, there is a single real eigenvalue of multiplicity 2.

   For $\omega > \omega_*$, the eigenvalues are complex conjugate, $|\lambda|_{min} = |\lambda|_{max}$.

3. Note that $\rho(B_\omega) \le ||B_\omega||_\infty$. This is consistent with a theorem proven in class.

4. The SOR scheme converges for $0 < \omega < 2$ since $\rho(B_\omega) < 1$ for those parameter values.

5. The optimal SOR method converges more rapidly than either Jacobi or Gauss-Seidel:

   $\rho(B_J) = 0.5$

   $\rho(B_{GS}) = 0.25$

   $\rho(B_{\omega*}) = \omega_* - 1 \sim 0.0718$