Number Systems

Base β

$$x = \pm(.a_1 a_2 a_3 \dots a_i \dots)_\beta \beta^e, \;\; 1 \le a_i < \beta$$

Chopping

$$\widetilde{x} = \pm(.a_1 a_2 a_3 \dots a_i)_\beta \beta^e$$

Rounding

$$\widetilde{x} = \begin{cases} \pm(.a_1 \dots a_i)_\beta \beta^e, & a_{i+1} < \frac{\beta}{2} \\ \pm[(.a_1 \dots a_i)_\beta \beta^e + (.0 \dots 1)_\beta \beta^e], & a_{i+1} \ge \frac{\beta}{2} \end{cases}$$

Error

Error: $e(\tilde{x}) = |x - \tilde{x}|$

Relative Error: $re(\tilde{x}) = \left|\frac{x-\tilde{x}}{x}\right|$

Linear Systems, $Ax = b$

THM: Given a matrix A, the following are equivalent

1. The Equation $Ax = b$ has a unique solution
2. A is invertible.
3. $\det(A) \neq 0$
4. $Ax = 0$ has a unique solution, $x = 0$
5. The columns of A are linearly independent
6. The eignevalues, λ, of A are non-zero.

Gaussian Elimination: $A = LU$
Gaussian Elimination with pivoting: $PA = LU$

Norms

Properties of Vector Norms:

$\|x\| \ge 0, \; \|x\| = 0 \Rightarrow x = 0$

$\|\lambda x\| = |\lambda| \; \|x\|, \quad \lambda$ scalar

$\|x + y\| \le \|x\| \; + \; \|y\|$

Vector Norms:

l_∞ : $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$

l_1 : $\|x\|_1 = \sum_{i=1}^n |x_i|$

l_2 : $\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$

Matrix Norm:

$$\|A\| = \max_{\|u\| \neq 0} \{\|Au\|/\|u\| : u \in \mathbf{R^n}\}$$

Properties of Matrix Norms:

$\|A\| \ge 0, \; \|A\| = 0 \Leftrightarrow A = 0$

$\|\lambda A\| = |\lambda| \; \|A\|$

$\|A + B\| \le \|A\| + \|B\|$

$\|Ax\| \le \|A\| \;\; \|x\|$

$\|AB\| \le \|A\| \;\; \|B\|$

Examples of Matrix Norms:

l_∞ Matrix Norm: $\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^n |a_{i,j}|$

l_1 Matrix Norm: $\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^n |a_{i,j}|$

l_2 Matrix Norm: $\|A\|_2 = \sqrt{\rho(A^* A)}$

Stability

Condition Number: $\kappa(A) = \|A^{-1}\| \; \|A\|$
Residual: $r = b - A\widetilde{x}$
THM:

$$\frac{1}{\kappa(A)} \frac{\|r\|}{\|b\|} \le \frac{\|e\|}{\|x\|} \le \kappa(A) \frac{\|r\|}{\|b\|}$$

Iterative Methods

$A = (L + D + U)$ where D is a diagonal matrix,L is lower triangular and U is upper triangular.

Jacobi: $Dx^{n+1} = -(L + U)x^n + b$

Gauss-Seidel: $Dx^{n+1} = -(Lx^{n+1} + Ux^n) + b$

SOR: $(D + \omega L)x^{n+1} = ((1 - \omega)D - \omega U)x^n + \omega b$

Root Finding Methods

Newton's Methods: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Secant Methods: $x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$

Error Bound for Bisection Method:

$$|\alpha - x_n| \le \left(\frac{1}{2}\right)^n |b_0 - a_0|$$

Polynomial Interpolation

Let f be defined on $[a,b]$; $x_0,\ x_1,\ldots,x_n$: $n+1$ distinct points in $[a,b]$. Let p_n be the interpolating polynomial of degree $\leq n$. Then

$$f(x)=p_n(x)+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\ldots(x-x_n)$$

for some $\xi\in[a,b]$.

Chebyshev Points

$$x_k=\cos{(2k+1)\pi 2n},\ k=0,1,\ldots,n-1$$

or

$$x_k=-\cos\pi kn,\ k=0,1,\ldots,n$$

Hermite Interpolation

Given f, x_0, $x_1,\ldots,x_n$: $n+1$ distinct points, the Hermite interpolating polynomial $p(x)$ ($\deg p\leq 2n+1$) is

$$p(x)=\sum_{i=0}^{n}\left(f(x_i)h_i(x)+f'(x_i)\tilde{h}_i(x)\right)$$

where

$$h_i(x)=\left(1-2(x-x_i)l_i'\right)l_i^2(x),\quad \tilde{h}_i(x)=(x-x_i)l_i^2(x)$$

If $f\in C^{(2n+2)}[a,b]$, $p(x)$ is the Hermite interpolating polynomial, then

$$f(x)=p(x)+\frac{f^{(2n+2)}(\xi)}{(2n+2)!}(x-x_0)^2\ldots(x-x_n)^2$$

Splines

Let $a=x_0<x_1<\ldots<x_{n-1}<x_n=b$. A spline of degree m is a function $S(x)$ which satisfies the following conditions:

1) For $x\in[x_i,x_{i+1}]$, $S(x)=S_i(x)$: polynomial of degree $\leq m$

2) $S^{(m-1)}(x)$ exists and is continuous at the interior points $x_1,\ldots,x_{n-1}$, i.e. $\lim_{x\to x_i^-}S_{i-1}^{(m-1)}(x)=\lim_{x\to x_i^+}S_i^{(m-1)}(x)$

Let f be defined on $[a,b]$, $a=x_0<x_1<\ldots<x_{n-1}<x_n=b$ and let S be the natural cubic spline interpolant of f. Then

$$1)\ |f(x)-S(x)|\leq\frac{5}{384}\max_{a\leq x\leq b}|f^{(4)}(x)|h^4$$

where $h=\max_i|x_{i+1}-x_i|$

$$\int_a^b\left(S''(x)\right)^2dx\leq\int_a^b\left(f''(x)\right)^2dx$$

Numerical Integration

$$\int_a^b f(x)dx\sim\sum_{i=0}^{n}c_if(x_i)$$

Trapezoid Rule:

$$T(h)=h\left(\frac{1}{2}f(x_0)+f(x_1)+\cdots+f(x_{n-1})+\frac{1}{2}f(x_n)\right)$$

Local Error Estimate:

$$\int_a^{a+h}f(x)dx=h\frac{f(a)+f(a+h)}{2}-\frac{h^3}{12}f''(\xi)$$

Global Error Estimate:

$$\int_a^b f(x)dx=T(h)-\frac{f''(\xi)}{12}h^2(b-a)$$

Simpson's Rule:

$$S(h)=h\left(\frac{1}{3}f(x_0)+\frac{4}{3}f(x_1)+\frac{2}{3}f(x_2)+\cdots\right.$$

$$\left.\cdots+\frac{2}{3}f(x_{n-2})+\frac{4}{3}f(x_{n-1})+\frac{1}{3}f(x_n)\right)$$

Error:

$$\int_a^b f(x)dx=S(h)-\frac{f^{(4)}(\xi)}{180}h^4(b-a)$$

Orthogonal Polynomials:

The *inner product* of two functions f and g on $[a,b]$ with the weighting function $w(x)$ is

$$\langle f,g\rangle=\int_a^b f(x)g(x)w(x)dx$$

Properties:

$$1)\ \langle f,f\rangle\geq 0,\langle f,f\rangle=||f||^2=0\Leftrightarrow f=0$$

$$2)\ \langle f,\alpha g+h\rangle=\alpha\langle f,g\rangle+\langle f,h\rangle$$

Gaussian Quadrature:

$$\int_{-1}^{1}f(x)dx\sim\sum_{i=1}^{n}c_if(x_i)$$

where $x_i,\ i=1,\ldots,n$ are roots of Legendre polynomial $P_n(x)$.

IVP for ODEs:

$$y'=f(t,y),\ y(a)=\alpha,\ a\leq t\leq b$$

Euler's Method:

$$u_{n+1}=u_n+hf(t_n,u_n),\ u_0=\alpha$$

Local Truncation Error: $\tau_n=\frac{h^2}{2}y''(\tilde{t}_n)$

Global Error:

$$|y_n-u_n|\leq\frac{hM}{2L}\left(\mathrm{e}^{L(t_n-a)}-1\right)$$

where L is a Lipschitz constant, $M = \max|y''(t)|$.

Modified Euler's Method:

$$k_1 = f(t_n, u_n),\ k_2 = f(t_n + h, u_n + hk_1)$$

$$u_{n+1} = u_n + \frac{h}{2}(k_1 + k_2)$$

4th Order Runge-Kutta Method:

$$k_1 = f(t_n, u_n),\ k_2 = f(t_n + \frac{h}{2}, u_n + \frac{h}{2}k_1)$$

$$k_3 = f(t_n + \frac{h}{2}, u_n + \frac{h}{2}k_2),\ k_4 = f(t_n + h, u_n + hk_3)$$

$$u_{n+1} = u_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Backward Euler's Method:

$$u_{n+1} = u_n + hf(u_{n+1})$$

System of ODEs:

$$y' = Ay,\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Exact Solution:

$$y(t) = \alpha_1(0)\,\mathrm{e}^{\lambda_1 t}\,p_1 + \alpha_2(0)\,\mathrm{e}^{\lambda_2 t}\,p_2$$

Forward Euler:

$$u_n = \alpha_1(0)\,(1 + h\lambda_1)^n\,p_1 + \alpha_2(0)\,(1 + h\lambda_2)^n\,p_2$$

Backward Euler:

$$u_n = \alpha_1(0)\left(\frac{1}{1 - h\lambda_1}\right)^n p_1 + \alpha_2(0)\left(\frac{1}{1 - h\lambda_2}\right)^n p_2$$

Multistep Methods:
General 2-Step Method:

$$\alpha_0 u_{n+1} + \alpha_1 u_n + \alpha_2 u_{n-1} = h\left[\beta_0 f(u_{n+1}) + \beta_1 f(u_n) + \beta_2 f(u_{n-1})\right]$$

Adams-Bashforth:

$$u_{n+1} = u_n + \frac{h}{2}\left[3f(u_n) - f(u_{n-1})\right]$$

Adams-Moulton

$$u_{n+1} = u_n + \frac{h}{12}\left[5f(u_{n+1}) + 8f(u_n) - f(u_{n-1})\right]$$

Leap-Frog

$$u_{n+1} = u_{n-1} + 2hf(u_n)$$

BDF: Backward Differentiation Formula — Gear's Method

$$\frac{3}{2}u_{n+1} - 2u_n + \frac{1}{2}u_{n-1} = hf(u_n)$$

Computing eigenvalues and eigenvectors

$$Ax = \lambda x, \quad x \neq 0$$

λ : eigenvalue

x : associated eigenvector

Power method:

Idea: $v,\ Av,\ A^2v,\ \ldots$

1. $v^{(0)}$: given, $||v^{(0)}||_2 = 1$
2. for $k = 1, 2, \ldots$
3. $w = Av^{(k-1)}$
4. $v^{(k)} = w/||w||_2$
5. $\lambda^{(k)} = \left(v^{(k)}\right)^T Av^{(k)}$

Inverse iteration:

Idea: apply power method to A^{-1}, $(A - \mu I)^{-1}$, μ: shift

1. $v^{(0)}$: given, $||v^{(0)}||_2 = 1$
2. for $k = 1, 2, \ldots$
3. solve $(A - \mu I)w = v^{(k-1)}$
4. $v^{(k)} = w/||w||_2$
5. $\lambda^{(k)} = \left(v^{(k)}\right)^T Av^{(k)}$

Rayleigh quotient iteration:

Idea: update μ

1. $v^{(0)}$: given, $||v^{(0)}||_2 = 1$, $\lambda^{(0)} = \left(v^{(0)}\right)^T Av^{(0)}$
2. for $k = 1, 2, \ldots$
3. solve $(A - \lambda^{(k-1)} I)w = v^{(k-1)}$
4. $v^{(k)} = w/||w||_2$
5. $\lambda^{(k)} = \left(v^{(k)}\right)^T Av^{(k)}$

Least Squares:

$A\vec{z} = \vec{b}$: $m \times n$ system with $m \geq n$

$A^T A\vec{z} = A^T\vec{b}$: normal equations