Number Systems

Base β

$$x = \pm(.a_1 a_2 a_3 \ldots a_i \ldots)_\beta \beta^e, \;\; 1 \le a_i < \beta$$

Chopping

$$\widetilde{x} = \pm(.a_1 a_2 a_3 \ldots a_i)_\beta \beta^e$$

Rounding

$$\widetilde{x} = \begin{cases} \pm(.a_1 \ldots a_i)_\beta \beta^e, & a_{i+1} < \frac{\beta}{2} \\ \pm[(.a_1 \ldots a_i)_\beta \beta^e + (.0 \ldots 1)_\beta \beta^e], & a_{i+1} \ge \frac{\beta}{2} \end{cases}$$

Error

Error: $e(\tilde{x}) = |x - \tilde{x}|$

Relative Error: $re(\tilde{x}) = \left|\frac{x-\tilde{x}}{x}\right|$

Linear Systems, $Ax = b$

THM: Given a matrix A, the following are equivalent

1. The Equation $Ax = b$ has a unique solution
2. A is invertible.
3. $\det(A) \neq 0$
4. $Ax = 0$ has a unique solution, $x = 0$
5. The columns of A are linearly independent
6. The eignevalues, λ, of A are non-zero.

Gaussian Elimination: $A = LU$
Gaussian Elimination with pivoting: $PA = LU$

Norms

Properties of Vector Norms:

$\|x\| \ge 0, \; \|x\| = 0 \Rightarrow x = 0$

$\|\lambda x\| = |\lambda| \; \|x\|, \quad \lambda$ scalar

$\|x + y\| \le \|x\| \; + \; \|y\|$

Vector Norms:

$l_\infty: \quad \|x\|_\infty = \max_{1 \le i \le n} |x_i|$

$l_1: \quad \|x\|_1 = \sum_{i=1}^n |x_i|$

$l_2: \quad \|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$

Matrix Norm:

$$\|A\| = \max_{\|u\| \neq 0} \{\|Au\| / \|u\| : u \in \mathbf{R}^n\}$$

Properties of Matrix Norms:

$\|A\| \ge 0, \; \|A\| = 0 \Leftrightarrow A = 0$

$\|\lambda A\| = |\lambda| \; \|A\|$

$\|A + B\| \le \|A\| + \|B\|$

$\|Ax\| \le \|A\| \;\; \|x\|$

$\|AB\| \le \|A\| \;\; \|B\|$

Examples of Matrix Norms:

l_∞ Matrix Norm: $\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^n |a_{i,j}|$

l_1 Matrix Norm: $\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^n |a_{i,j}|$

l_2 Matrix Norm: $\|A\|_2 = \sqrt{\rho(A^* A)}$

Stability

Condition Number: $\kappa(A) = \|A^{-1}\| \; \|A\|$
Residual: $r = b - A\widetilde{x}$
THM:

$$\frac{1}{\kappa(A)} \frac{\|r\|}{\|b\|} \le \frac{\|e\|}{\|x\|} \le \kappa(A) \frac{\|r\|}{\|b\|}$$

Root Finding Methods

Newton's Methods: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Secant Methods: $x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$

Error Bound for Bisection Method:

$$|\alpha - x_n| \le \left(\frac{1}{2}\right)^n |b_0 - a_0|$$