Elementary Matrices

Let us introduce the following matrices. We call the **elementary matrix** the matrix

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & 0 & \\ & & 1 & & & & \\ \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot \\ & & & & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \quad (i) \tag{1}$$

which is different from the identity matrix only by that it has the entry α on the ith position of the main diagonal, $1 \leq i \leq n$, where $\alpha \neq 0$ is an arbitrary number.

We also call **elementary λ-matrix**, the matrix

$$\begin{pmatrix} 1 & & & & \cdot & & \\ & \ddots & & & \vdots & & \\ \cdot & \cdots & 1 & \cdots & \phi(\lambda) & \cdots & \cdot \\ & & & \ddots & \vdots & & \\ & & & & 1 & & \\ & & & & \vdots & \ddots & \\ & & & & \cdot & & 1 \end{pmatrix} \quad (i)$$

$$(j)$$

that is different from the identity matrix by that it has an arbitrary polynomial $\phi(\lambda)$ at the intersection of ith row and jth column, $1 \leq i \leq n$, $1 \leq j \leq n$, and $i \neq j$.

Any elementary transformation of a matrix A is equivalent to multiplication of this matrix from the left or right by some elementary matrix.

Indeed, the following four statements are true:

- multiplication of the matrix A by matrix (1) from the left is equivalent to multiplication of ith row of matrix A by number α;
- multiplication of the matrix A by matrix (1) from the right is equivalent to multiplication of ith column of matrix A by number α;

- multiplication of the matrix A by matrix (2) from the left is equivalent to adding jth row of matrix A multiplied by $\phi(\lambda)$ to its ith row;
- multiplication of the matrix A by matrix (2) from the right is equivalent to adding ith column of matrix A multiplied by $\phi(\lambda)$ to its jth column;

Denote by E_{ij} a matrix obtained from the unit matrix by interchanging ith and jth rows.

$$E_{ij} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ (i) \\ \\ (j) \\ \\ \\ \end{matrix} \tag{3}$$

The following statements are true:

- multiplication of the matrix A by matrix (3) from the left is equivalent to interchanging of ith and jth rows of matrix A;
- multiplication of the matrix A by matrix (3) from the right is equivalent to interchanging of ith and jth columns of matrix A;

`EXAMPLE`
Multiply 2nd row by α.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Elementary matrix that we need to use is

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we multiply matrix A by E from the left, i.e.

$$EA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

`EXAMPLE` Interchange 2nd and 3rd rows of matrix A

$$E_{23}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

`EXAMPLE`

Consider again matrix A and let's eliminate variable x_1 from the 2nd and 3rd equations/rows using Gaussian elimination. Then we will identify which elementary matrices were used and find total transformation (matrix) which reduces A to the equivalent matrix which has entries in the first column below the main diagonal zero.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \sim \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & a_{23} - \frac{a_{21}}{a_{11}} a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where we subtracted 1st row multiplied by $\frac{a_{21}}{a_{11}}$ from the 2nd row, i.e.

$$\text{2nd row} - \tfrac{a_{21}}{a_{11}} \cdot \text{1st row} \longrightarrow \text{2nd row} \Rightarrow m_{21} = \tfrac{a_{21}}{a_{11}}$$

$$i = 2 \qquad\qquad j = 1$$

and elementary matrix is

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (i = 2)$$

$$(j = 1)$$

Check this with elementary matrix.

$$E_1 \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & -\frac{a_{21}}{a_{11}} a_{12} + a_{22} & -\frac{a_{21}}{a_{11}} a_{13} + a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \text{OK}$$

Now, let us eliminate variable x_1 from the 3rd equation

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \sim \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} \\ 0 & \tilde{a}_{32} & \tilde{a}_{33} \end{pmatrix}$$

where $\tilde{}$ indicates transformed entries.

3rd row $-\ \frac{a_{31}}{a_{11}} \cdot$ 1st row $\longrightarrow$ 3rd row $\Rightarrow m_{31} = \frac{a_{31}}{a_{11}}$

$i = 3 \qquad\qquad j = 1$

and elementary matrix is

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \end{pmatrix} \qquad (i = 3)$$

$(j = 1)$

The resulting total transformation matrix is $B = E_2 E_1$ (<u>note</u>: reversed order) and

$$BA = E_2 E_1 A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} \\ 0 & \tilde{a}_{32} & \tilde{a}_{33} \end{pmatrix}$$

`EXAMPLE`

$$\begin{cases} 2x_1 - x_2 = 1 \\ -x_1 + 2x_2 - x_3 = 0 \\ -x_2 + 2x_3 = 1 \end{cases}$$

Denote

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Then the augmented matrix $A|b$ is

$$\begin{pmatrix} 2 & -1 & 0 & \vdots & 1 \\ -1 & 2 & -1 & \vdots & 0 \\ 0 & -1 & 2 & \vdots & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 0 & \vdots & 1 \\ 0 & \frac{3}{2} & -1 & \vdots & \frac{1}{2} \\ 0 & -1 & 2 & \vdots & 1 \end{pmatrix} \sim$$

$$\text{2nd row} + \tfrac{1}{2} \cdot \text{1st row} \to \text{2rd row} \Rightarrow m_{21} = -\tfrac{1}{2}$$

$$i = 2 \qquad\qquad j = 1$$

and elementary matrix is

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (i = 2)$$

$$(j = 1)$$

Here $m_{31} = 0$ since $a_{31} = 0$ in the given matrix.

Now we eliminate x_2 from the 3rd equation

$$\begin{pmatrix} 2 & -1 & 0 & \vdots & 1 \\ 0 & \frac{3}{2} & -1 & \vdots & \frac{1}{2} \\ 0 & 0 & \frac{4}{3} & \vdots & \frac{4}{3} \end{pmatrix}$$

$$\text{3rd row} + \tfrac{2}{3} \cdot \text{2nd row} \to \text{3rd row} \Rightarrow m_{32} = -\tfrac{2}{3}$$

$$i = 3 \qquad\qquad j = 2$$

and elementary matrix is

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix} \qquad (i = 3)$$

$$(j = 2)$$

Therefore, the reduced matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix} \equiv \tilde{A} = E_2 E_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 \end{pmatrix} A \quad \Rightarrow \tilde{A} = BA,$$

where $B = E_2 E_1$ is the transformation matrix.

<u>Check:</u>

$$BA = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix} = \tilde{A} \quad \text{OK}$$