1 Exact Equations and Integrating Factors

Def Given $Mdx + Ndy = 0$ with $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, i.e. not exact. If

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact, i.e.

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

then $\mu(x,y)$ is called an **integrating factor**.

In example with equation (A), $\frac{1}{t}$ is an integrating factor, in (B) $\frac{1}{ty}$ is an integrating factor.

How do we find integrating factors? What are the conditions on $\mu(x,y)$ so as to make $\mu(x,y)$ an integrating factor?

ANS

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}; \qquad \mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x}$$

or

$$\boxed{M}\frac{\partial \mu}{\partial y} - \boxed{N}\frac{\partial \mu}{\partial x} + \mu\boxed{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)} = 0 \qquad \text{a PDE}$$

Is there an integrating factor $\mu(x)$, function of x only?

$$-N\frac{d\mu}{dx} + \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 0$$

or

$$\frac{d\mu}{\mu} = \boxed{\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}dx$$

ANS YES if $\boxed{\;...\;}$ is a function of x only.

Is there an integrating factor $\mu(y)$, function of y only?

$$M\frac{d\mu}{dy} + \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 0$$

or

$$\frac{d\mu}{\mu} = \boxed{\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}dy$$

<u>ANS</u> YES if $\boxed{\ \ldots\ }$ is a function of y only.

$\therefore$ If $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

see if (a) $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ is a function of x only.

YES $\Rightarrow \mu = \mu(x)$ only. Solve a separable DE for $\mu(x)$.

see if (b) $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$ is a function of y only.

YES $\Rightarrow \mu = \mu(y)$ only. Solve a separable DE for $\mu(y)$.

<u>Example</u>

$$\frac{dy}{dx} = -\frac{x^2 + y^2 + x}{xy}, \quad y(2) = -1. \text{ Find } y(1).$$

$$(x^2 + y^2 + x)dx + xydy = 0$$

$$\frac{\partial M}{\partial y} = 2y; \quad \frac{\partial N}{\partial x} = y \quad \text{not exact}$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{x^2 + y^2 + x}(y - 2y) \quad \text{not a function of } y$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}(2y - y) = \frac{1}{x} \quad \text{Good!}$$

$$\frac{d\mu}{\mu} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx$$

$$\frac{d\mu}{\mu} = \frac{1}{x}dx \Rightarrow \ln|\mu| = \ln|x| + \ln\tilde{C} \quad \Rightarrow \quad \mu = Cx \text{ is an integrating factor}$$

Now DE with $\mu = x$

$$\Rightarrow (x^3 + xy^2 + x^2)dx + x^2y dy = 0 \quad \text{is supposed to be exact}$$

CHECK:

$$\frac{\partial M^*}{\partial y} = 2xy = \frac{\partial N^*}{\partial x} \quad \text{EXACT}$$

$$\left.\begin{array}{l} f(x,y) = \dfrac{x^4}{4} + \dfrac{x^2y^2}{2} + \dfrac{x^3}{3} + \theta(y) \\ f(x,y) = \qquad \dfrac{x^2y^2}{2} \qquad + \psi(x) \end{array}\right. \Rightarrow f(x,y) = \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2y^2}{2}$$

$$\Rightarrow d\left[\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2y^2}{2}\right] = 0 \quad \Rightarrow \quad \boxed{\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2y^2}{2} = C}$$

$$3x^4 + 4x^3 + 6x^2y^2 = C_1 \quad \text{I.C.} \Rightarrow 3(2)^4 + 4(2)^3 + 6(2)^2(-1)^2 = 104 = C_1$$

$$3x^4 + 4x^3 + 6x^2y^2 = 104 \quad \text{to find } y(1) \text{ solve for } y$$

$$y = -\sqrt{\frac{104 - x^4 - 4x^3}{6x^2}} \quad \underline{\text{note the choice of } - \text{ in } \sqrt{\ }}$$

$$y(1) = -\sqrt{\frac{104 - 3 - 4}{6}} = -4.02$$