#17 S 14.2

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} \overset{\text{conjugate factor}}{=} \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{(\sqrt{x^2+y^2+1}-1)(\sqrt{x^2+y^2+1}+1)} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{\cancel{(x^2+y^2)}(\sqrt{x^2+y^2+1}+1)}{\cancel{(x^2+y^2+1)-1}} = 2 \quad \text{(by continuity)}$$

Now, let's prove that the limit $= 2$.

$\forall \varepsilon > 0 \;\; \exists \delta = \delta(\varepsilon) > 0: \quad 0 < \sqrt{(x-0)^2+(y-0)^2} < \delta \Rightarrow$

$\Rightarrow \left| \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} - 2 \right| < \varepsilon$ $\qquad x^2+y^2 < \delta^2$

Need to express δ as a function of ε.

$$\left| \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} - 2 \right| = \left| \frac{x^2+y^2 - 2(\sqrt{x^2+y^2+1}-1)}{\sqrt{x^2+y^2+1}-1} \right| =$$

$$= \frac{x^2+y^2-2\sqrt{x^2+y^2+1}+2}{\sqrt{x^2+y^2+1}-1} = \left| \frac{x^2+y^2+1-2\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}-1} \right| =$$

$$= \frac{(\sqrt{x^2+y^2+1}-1)^2}{|\sqrt{x^2+y^2+1}-1|} = |\sqrt{x^2+y^2+1}-1| = \frac{(\sqrt{x^2+y^2+1}-1)(\sqrt{x^2+y^2+1}+1)}{\sqrt{x^2+y^2+1}+1} =$$

$$= \frac{(x^2+y^2+\cancel{1})-\cancel{1}}{\sqrt{x^2+y^2+1}+1} < \frac{x^2+y^2}{2} < \frac{\delta^2}{2} = \varepsilon \Rightarrow \text{take } \frac{\delta^2}{2} = \varepsilon \text{ or}$$

$$\delta^2 = 2\varepsilon$$

$$\underbrace{\sqrt{x^2+y^2+1}}_{>0} + 1 > 2 \Rightarrow \frac{1}{\sqrt{x^2+y^2+1}+1} < \frac{1}{2} \qquad \delta = \sqrt{2\varepsilon} = \delta(\varepsilon)$$

We showed that for any $\varepsilon > 0$ there is

$\delta = \sqrt{2\varepsilon} > 0$: $\sqrt{(x-0)^2 + (y-0)^2} < \delta$, i.e. $x^2 + y^2 < \delta$

we have $\left| \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1} - 1} - 2 \right| < \varepsilon$ ∎