**Math 571 - Functional Analysis I - Fall 2017**

**Homework 10**

Due: **Friday, November 10, 2017**

1. (# 8, Section 3.2) Show that in an inner product space, $x \perp y$ if and only if $||x + \alpha y|| \geq ||x||$ for all scalars $\alpha$.

2. (# 9, Section 3.2) Let $V$ be the vector space of all continuous complex-valued functions on $J = [a, b]$. Let $X_1 = (V, ||.||_\infty)$, where $||x||_\infty = \max_{t \in J} |x(t)|$; and let $X_2 = (V, ||.||_2)$, where

$$||x||_2 = \langle x, x \rangle^{1/2}, \quad \langle x, y \rangle = \int_a^b x(t)\overline{y(t)}dt.$$

Show that the identity mapping $x \to x$ of $X_1$ onto $X_2$ is continuous. (It is not a homeomorphism. $X_2$ is not complete.)

3. (# 1, Section 3.3) Let $H$ be a Hilbert space, $M \subset H$ a convex subset, and $(x_n)$ a sequence in $M$ such that $||x_n|| \to d$, where $d = \inf_{x \in M} ||x||$. Show that $(x_n)$ converges in $H$. Give an illustrative example in $\mathbb{R}^2$ or $\mathbb{R}^3$.

4. (# 3, Section 3.3)

   (a) Show that the vector space $X$ of all real-valued continuous functions on $[-1, 1]$ is the direct sum of the set of all even continuous functions and the set of all odd continuous functions on $[-1, 1]$.

   (b) Give examples of representations of $\mathbb{R}^3$ as a direct sum (i) of a subspace and its orthogonal complement, (ii) of any complementary pair of subspaces.

5. (# 6, Section 3.3) Show that $Y = \{x : x = (\xi_j) \in l^2, \ \xi_{2n} = 0, \ n \in \mathbb{N}\}$ is a closed subspace of $l^2$ and find $Y^\perp$. What is $Y^\perp$ if $Y = \text{span}\{e_1, \ldots, e_n\} \subset l^2$, where $e_j = (\delta_{jk})$?

**Suggested problems:**

6. (# 8, Section 3.3) Show that the annihilator $M^\perp$ of a set $M \neq$ in an inner product space $X$ is closed subspace of $X$.

7. (# 4, Section 3.4) Give an example of an $x \in l^2$ such that we have strict inequality in Bessel inequality

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq ||x||^2.$$