Math 571 - Functional Analysis I - Fall 2017

Homework 11

Due: **Friday, November 17, 2017**

1. (# 5, Section 3.4) If (e_k) is an orthonormal sequence in an inner product space X, and $x \in X$, show that $x - y$ with y given by

$$y = \sum_{k=1}^{n} \alpha_k e_k, \quad \alpha_k = \langle x, e_k \rangle$$

is orthogonal to the subspace $Y_n = \operatorname{span}\{e_1, \ldots, e_n\}$.

2. (# 6, Section 3.4) (**Minimum property of Fourier coefficients**) Let $\{e_1, \ldots, e_n\}$ be an orthonormal set in an inner product space X, where n is fixed. Let $x \in X$ be any fixed element and $y = \beta_1 e_1 + \ldots + \beta_n e_n$. Then $||x - y||$ depends on $\beta_1, \ldots, \beta_n$. Show by direct calculation that $||x - y||$ is minimum if and only if $\beta_j = \langle x, e_j \rangle$, where $j = 1, \ldots, n$.

3. (# 9, Section 3.4) Orthonormalize the first three terms of the sequence $(x_0, x_1, x_2, \ldots)$, where $x_j(t) = t^j$ on the interval $[-1, 1]$, where

$$\langle x, y \rangle = \int_{-1}^{1} x(t) y(t) dt.$$

4. (# 3, Section 3.5) Illustrate with an example that a convergent series $\sum \langle x, e_k \rangle e_k$ need not have the sum x.

5. (# 4, Section 3.5) If (x_j) is a sequence in an inner product space X such that the series $||x_1|| + ||x_2|| + \ldots$ converges, show that (s_n) is a Cauchy sequence, where $s_n = x_1 + \ldots + x_n$.

6. (# 6, Section 3.5) Let (e_j) be an orthonormal sequence in a Hilbert space H. Show that if

$$x = \sum_{j=1}^{\infty} \alpha_j e_j, \quad y = \sum_{j=1}^{\infty} \beta_j e_j,$$

then

$$\langle x, y \rangle = \sum_{j=1}^{\infty} \alpha_j \overline{\beta}_j,$$

the series being absolutely convergent.

7. (# 1, Section 3.6) If F is an orthonormal basis in an inner product space X, can we represent every $x \in X$ as a linear combination of elements of F? (By definition, a linear combination consists of finitely many terms.)