Math 571 - Functional Analysis I - Fall 2017

Homework 12

Due: **Friday, November 17, 2017**

1. (# 4, Section 3.6) Derive from the Parseval relation

$$\sum_k |\langle x, e_k\rangle|^2 = ||x||^2$$

the following formula (which is often called the *Parseval relation*).

$$\langle x, y\rangle = \sum_k \langle x, e_k\rangle \overline{\langle y, e_k\rangle}.$$

2. (# 5, Section 3.6) Show that an orthonormal family (e_k), $k \in I$, in an Hilbert space H is total if and only if the relation in Prob. 1 holds for every x and y in H.

3. (# 1, Section 3.7) Show that the Legendre differential equation can be written

$$[(1-t^2)P_n']' = -n(n+1)P_n.$$

Multiply this by P_m. Multiply the corresponding equation for P_m by $-P_n$ and add the two equations. Integrating the resulting equation from -1 to 1, show that (P_n) is an orthogonal sequence in the space $L^2[-1,1]$.

4. (# 9, Section 3.7) Solve the differential equation $y'' + (2n+1-t^2)y = 0$ in terms of the Hermite polynomials.

5. (# 15, Section 3.7) Show that the functions

$$e_n(t) = \mathrm{e}^{-t/2}\, L_n(t)$$

where $L_n(t)$ is the Laguerre polynomial of order n, constitute an orthogonal sequence in the space $L^2[0,+\infty)$.

6. (# 2, Section 3.8) (**Space** l^2) Show that every bounded linear functional f on l^2 can be represented in the form

$$f(x) = \sum_{j=1}^{\infty} \xi_j \bar{\zeta}_j$$

where $z = (\zeta_j) \in l^2$.

7. (# 3, Section 3.8) If z is any fixed element of an inner product space X, show that $f(x) = \langle x, z\rangle$ defines a bounded linear functional f on X, of norm $||z||$.