Math 539 - Theory of Ordinary Differential Equations - Fall 2017

Homework 1

Due: **Friday, September 1, 2017**

Consider the differential equation

$$Lu = a_0(x)u'' + a_1(x)u' + a_2(x)u = f(x), \quad a < x < b \tag{1}$$

where a_0, a_1, a_2 are continuous and f is piecewise continuous in $a \le x \le b$. We require u to satisfy the two homogenous boundary conditions

$$B_1(u) = 0: \quad \alpha_{11}u(a) + \alpha_{12}u'(a) + \beta_{11}u(b) + \beta_{12}u'(b) = 0, \tag{2}$$

$$B_2(u) = 0: \quad \alpha_{21}u(a) + \alpha_{22}u'(a) + \beta_{21}u(b) + \beta_{22}u'(b) = 0, \tag{3}$$

where the coefficients α_{ij}, β_{ij} are real numbers.

To ensure that the boundary conditions are distinct, we assume that row vectors $(\alpha_{11}, \alpha_{12}, \beta_{11}, \beta_{12})$ and $(\alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22})$ are independent.

If $\beta_{11} = \beta_{12} = \alpha_{21} = \alpha_{22} = 0$, the conditions are *unmixed* (one condition per end point):

$$\alpha_{11}u(a) + \alpha_{12}u'(a) = 0 \tag{4}$$

$$\beta_{21}u(b) + \beta_{22}u'(b) = 0 \tag{5}$$

1. Let $L = L^*$, and let the boundary conditions be of type (2), (3). Show that the necessary and sufficient conditions for the system (1) - (3) to be self-adjoint is $a_0(a)p_{34} = a_0(b)p_{12}$, where

   $$p_{12} = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}, \quad p_{34} = \begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{vmatrix}$$

   These conditions are obviously satisfied if the boundary conditions are unmixed.

   As another special case, show that the "periodic" boundary conditions $u(a) = u(b)$, $a_0(a)u'(a) = a_0(b)u'(b)$ lead to a self-adjoint system when associated with a formally self-adjoint operator.

2. Let $L = (d^2/dx^2) + 4(d/dx) - 3$. Show that $L^* = (d^2/dx^2) - 4(d/dx) - 3$, $J(u, v) = vu' - uv' + 4uv$. If we are given the boundary conditions $u'(a) + 4u(a) = 0$ and $u'(b) + 4u(b) = 0$, show that the adjoint boundary conditions are $v'(a) = 0$ and $b'(b) = 0$.

3. For an nth-order linear differential operator, the concepts of self-adjointness and adjoint operators are defined in a manner similar to that used for second-order operators. Find/derive the formal self-adjoint of such an nth-order operator. Show that an operator of odd order cannot be formally self-adjoint.

   Consider $L = d^4/dx^4$ and the four boundary conditions $u(0) = 0$, $u(1) = 0$, $u'(0) = 0$, $u'(1) = 0$. Show that this system is self-adjoint.

   Let L=d/dx with the boundary condition $u(0) = \alpha u(1)$. Find L^* and D_B^*.

4. Prove that, if L is a second-order operator, there exists $s(x)$ so that $L_1 = sL$ is formally self-adjoint.

5. Show that $(L^*)^* = L$ and $(D_B^*)^* = D_B$, for systems of type (1) - (3).