Math 539 - Theory of Ordinary Differential Equations - Fall 2017

Homework 2

Due: **Monday, September 11, 2017**

1. Verify the statements about each of the following two point boundary value problems, and construct either the unique solution or the general form of a nonunique solution, when it exists:

   (i) There is a unique solution to

   $$u'' = 0 \quad x \in (0,1)\,, \quad \text{with } u'(0) = 1\,, \quad u(1) = 3\,.$$

   (ii) There are infinitely many solutions to

   $$u'' = 0 \quad x \in (0,1)\,, \quad \text{with } u'(0) = 1\,, \quad u'(1) = 1\,.$$

   Also, there are infinitely many solutions to

   $$u'' + \pi^2 u = 0 \quad x \in (0,1)\,, \quad \text{with } u(0) = 0\,, \quad u(1) = 0\,.$$

   (iii) There is no solution to

   $$u'' = 0 \quad x \in (0,1)\,, \quad \text{with } u'(0) = 1 \quad u'(1) = 0\,.$$

2. For the second order differential operator

   $$L = x\frac{d^2}{dx^2} + \frac{d}{dx} + \sin x \quad x \in (1,2)$$

   find the adjoint operator $\mathcal{L}^* = \{L^*, D_B^*\}$ for the following sets of boundary data:

   $$\begin{array}{cl} \text{(i)} & B_1 u = u(1) + u'(1), \quad B_2 u = u'(2) \\ \text{(ii)} & B_1 u = u(1) + u(2), \quad B_2 u = u'(1) + u'(2). \end{array}$$

   Is the boundary value problem self-adjoint in either case, i.e., does $\mathcal{L} = \mathcal{L}^*$?

3. For the boundary value problem

   $$Lu \equiv u'' + u' = f(x) \quad x \in (0,1)$$

   $$B_1 u \equiv u'(0) + au(0) = c_1 \quad B_2 u \equiv u'(1) = c_2$$

   where a is a constant, is $\mathcal{L}^* = \{L^*, D_B^*\}$ self-adjoint, is L formally self-adjoint, or neither? Construct the adjoint differential operator and the adjoint boundary conditions by forming Green's identity $< v, Lu >= [J(u,v)]_a^b + < L^*v, u >$.

4. The inner product of two functions $u(x)$ and $v(x)$ with a weight function $w(x)$ for $x \in [a, b]$ is

$$< u, v >= \int_a^b uv\, w\, dx.$$

(a) The general second order linear differential operator is $Lu = a_0(x)\frac{d^2u}{dx^2} + a_1(x)\frac{du}{dx} + a_2(x)u$, where the functions $a_i(x)$ are smooth and $a_0(x) \neq 0$ for $x \in (a, b)$. Form Green's identity $< v, Lu >= [J(u, v)]_a^b + < L^*v, u >$ to construct L^* and $J(u, v)$ when the inner product includes a non-trivial weight function $w(x)$. Find the conditions on $a_i(x)$ for which $L = L^*$. You should find that it is $wa_1 = (wa_0)'$. Use this to simplify the expression for L when $L = L^*$, and by renaming the $a_i(x)$ show that any second order linear differential operator that is formally self-adjoint can be written as

$$Lu = \frac{-1}{w(x)}\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u. \tag{1}$$

Show that the conjunct is then

$$J(u, v) = p(x)(uv' - vu').$$

(b) The formally self-adjoint differential operator L of (1) is called a Sturm-Liouville operator. Show that it gives a self-adjoint operator $\mathcal{L} = \{L, D_B\}$ for the boundary value problem with the following two sets of boundary conditions:

(i) separated (or unmixed) boundary conditions

$$\begin{aligned} B_1u &= \alpha_1u(a) + \beta_1u'(a) \\ B_2u &= \alpha_2u(b) + \beta_2u'(b) \end{aligned}$$

(ii) periodic boundary conditions, provided $p(x)$ is also periodic so that $p(a) = p(b)$,

$$B_1u = u(a) - u(b) \quad \text{and} \quad B_2u = u'(a) - u'(b).$$

5. If $u_1, \ldots, u_n$ are any solutions of the n^{th} order linear homogeneous ordinary differential equation

$$Lu = a_0u^{(n)} + a_1u^{(n-1)} + \ldots + a_{n-1}u^{(1)} + a_nu = 0$$

briefly show that their Wronskian W satisfies 'Abel's formula'

$$W(u_1, \ldots, u_n)(x) = C\exp\left(-\int^x \frac{a_1(s)}{a_0(s)}ds\right)$$

where $a_0(x) \neq 0$ for $x \in I$ and C is constant. (You can find this result in, for example, the book on ODE's by Boyce and DiPrima.) Explain why for a given choice of the n solutions either $W = 0$ for all $x \in I$ or $W \neq 0$ for all $x \in I$.

Show that when $n = 2$ and L is formally self-adjoint Abel's formula becomes $W(x) = C/a_0(x)$. Use this to simplify the expression (3.4) for the Green's function $G(x, \xi)$ as given in Section 3.2 of the notes when the boundary conditions are separated (i.e. unmixed), and verify that in this case $G(x, \xi)$ is symmetric in x and ξ. What result in the theory tells you to expect that $G(x, \xi)$ is symmetric.