Math 539 - Theory of Ordinary Differential Equations - Fall 2017

Homework 3

Due: **Wednesday, September 20, 2017**

1. Use the Green's function $G(x, \xi)$ of equation (3.4) in the notes, which is for an arbitrary second order linear differential operator with separated (i.e. unmixed) boundary conditions, to show that the expression (3.9)

$$u(x) = \int_a^b G(x, \xi) f(\xi)\, d\xi$$

satisfies the boundary value problem (3.1) with *homogeneous* boundary conditions. (This proves the result of Section 3.3 of the notes for a second order differential operator and separated boundary conditions.)

2. For the boundary value problems given by

$$Lu = u'' = f(x) \qquad x \in (0, 1)$$

with the two different sets of boundary conditions given by

$$\begin{array}{lll} \text{(i)} & B_1 u = u(0) = c_1, & B_2 u = u(1) = c_2, \\ \text{(ii)} & B_1 u = u(0) = c_1, & B_2 u = u'(1) = c_2, \end{array}$$

show in both cases that a Green's function exists, and find it.

3. For the boundary value problem

$$\begin{aligned} Lu = u'' - k^2 u = f(x) \qquad & x \in (0, 1), \\ B_1 u = u(0) - u'(0) = c_1, \qquad & B_2 u = u(1) = c_2, \end{aligned}$$

show that a Green's function exists for all $k \geq 0$. The cases $k > 0$ and $k = 0$ need to be considered separately, do so, and explain why? Find the Green's function when $k > 0$. What happens to this Green's function as $k \to 0$? Why should you expect your expression for the Green's function $G(x, \xi)$ to be symmetric in x and ξ for all $k \geq 0$?

4. For the boundary value problem

$$\begin{aligned} Lu = u'' + \alpha^2 u = f(x) \qquad & x \in (0, 1), \\ B_1 u = u(0) - u(1) = 0 \qquad & B_2 u = u'(0) - u'(1) = 0, \end{aligned}$$

show that a Green's function exists for all $\alpha \neq 2n\pi$ ($n = 0, \pm 1, \pm 2, \ldots$). Find the Green's function when it exists (note that the boundary conditions are periodic, and therefore not separated). What happens to the Green's function as $\alpha \to 2n\pi$? why should you expect the Green's function $G(x, \xi)$ to be symmetric in x and ξ?

5. If $\alpha > 0$ is a real constant find the Green's function $G(x,\xi)$ for the problem

$$-\frac{d^2G}{dx^2} + \alpha^2 G = \delta(x-\xi) \qquad x,\xi \in (0,1)$$

$$\frac{dG}{dx}(0,\xi) = \frac{dG}{dx}(1,\xi) = 0$$

Why should you expect your expresion for $G(x,\xi)$ to be symmetric in x and ξ? What happens to the Green's function as $\alpha \to 0$?

Show that when $\alpha = 0$ you can not expect a Green's function to exist. Find a physical interpretation of this (non-existence) result in terms of one-dimensional heat conduction.