Math 539 - Theory of Ordinary Differential Equations - Fall 2017

<u>**Homework 4**</u>

Due: **Wednesday, October 11, 2017**

1. For the boundary value problem

$$Lu = u'' - u = f(x) \qquad x \in (0,1)$$
$$B_1u = u(1) - 2u(0) = c_1 \qquad B_2u = u'(1) = c_2$$

show that a Green's function exists, and find it. Note that the boundary conditions are not separated. Why would you expect that your expression for $G(x,\xi)$ may not be symmetric in x and ξ?

2. In problems 2 and 4 from HW # 3, you have constructed the Green's function for a two point boundary value problem. Now, find the solution to the boundary value problem

$$Lu = f, \quad B_iu = c_i, \quad i = 1, 2, \tag{1}$$

with inhomogeneous data (f, c_i). Write this in the form $u = u_f + u_c$, where u_f is the part of the solution that is the response to f alone (i.e., (1) with $c_i = 0$) and u_c is the response to $c_i, i = 1, 2$, alone (i.e., (1) with $f = 0$). In each case, leave the expression for the response to the inhomogeneity f in the differential equation in terms of the symbol $G(x,\xi)$ but find the response to the inhomogeneous boundary data c_1, c_2 explicitly.

Notice that in problem 4 from HW # 3 the boundary conditions are periodic. Periodic boundary conditions are almost always homogeneous in practice (i.e., $c_i = 0$), but you can still compute or find the response for periodic boundary conditions that are inhomogeneous ($c_i \neq 0$). In this case you may find the algebra simplest if you look for u_c to be a linear combination of $\sin\alpha(\frac{1}{2} - x)$ and $\cos\alpha(\frac{1}{2} - x)$.

3. We want to know what the influence of a weight function $w(x) \neq 0$ is on a two point boundary value problem with a given but general differential operator L and boundary conditions B_i. The weight function appears in the inner product, which is $< u, v >= \int_a^b uv\, w\, dx$.

(a) Repeat the reasoning in Section 3.4 of the lecture notes by applying Green's identity to $< G^*(x,\eta), LG(x,\xi) >$ to show that the relation (3.12) between the Green's function $G(x,\xi)$ and its adjoint $G^*(x,\xi)$ becomes $w(\xi)G^*(\xi,x) = w(x)G(x,\xi)$.

(b) Repeat the arguments in Section 3.5 of the lecture notes to show that the solution of the two point boundary value problem $Lu = f(x)$ with boundary conditions $B_i(u) = c_i \neq 0$ $(i = 1, \dots, n)$ is now such that

$$u(x) = \int_a^b G(x,\xi)f(\xi)\,d\xi - [J_\xi(u(\xi), \frac{G(x,\xi)}{w(\xi)}]_{\xi=a}^{\xi=b}. \tag{2}$$

This is the generalization of equation (3.14) of the notes when there is a general or non-trivial weight function $w(x)$. Note that Green's identity still takes the form $< v, Lu >=< L^*v, u > +[J(u, v)]_{x=a}^{x=b}$.

(c) For the general second order operator

$$Lu = a_0u'' + a_1u' + a_2u$$

construct L^* and $J(u, v)$. These must be the same as in the first steps of Examples 1, question 6(a) after multiplying a_0 and a_1 there by w. Hence show that that contribution from the boundary data given by the last term on the right hand side of (2) is in fact independent of the choice of w. This is true for any order n, and we should expect this result since both the (direct) Green's function $G(x, \xi)$ and the solution $u(x)$ of the boundary value problem are defined independently of w and therefore can not depend on it. However, for the adjoint problem, L^*, B_i^*, and $G^*(x, \xi)$ *do* all depend on w.

4. Show that the general second order differential operator

$$Lu = a_0u'' + a_1u' + a_2u$$

(where each a_i is a function of x and $a_0(x) \neq 0$) can be transformed to the Sturm-Liouville operator

$$Lu = \frac{-1}{w(x)}\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u,$$

by setting

$$a_0 = -\frac{p}{w}, \qquad a_1 = -\frac{p'}{w}, \qquad a_2 = q.$$

Since the Sturm-Liouville differential operator is formally self-adjoint with the inner product $< u, v >= \int_a^b uvw\,dx$, show by solving for p and w that any second order differential operator can be transformed to one that is formally self-adjoint provided we are free to choose the weight function w (up to an arbitrary multiplicative constant) as

$$w = -\frac{1}{a_0}\exp\left(\int \frac{a_1}{a_0}\,dx\right), \qquad \text{and then} \quad p = \exp\left(\int \frac{a_1}{a_0}\,dx\right), \qquad q = a_2.$$

5. In problem 3 from HW # 2, we constructed the adjoint operators for the problem

$$Lu \equiv u'' + u' = f(x) \quad x \in (0, 1)$$

$$B_1u \equiv u'(0) + au(0) = c_1 \quad B_2u \equiv u'(1) = c_2$$

when the weight is $w = 1$. Construct the Green's function in this case, when $a \neq 0$.

Suppose now that you are free to choose the weight function w. Find the choice of w that makes L formally self-adjoint, $L = L^*$. Construct it from first principles using Green's identity, and then check your answer against the result of problem 3 from HW # 2. Using Green's identity, show also that the boundary operators are such that $B_i = B_i^*$, so that the boundary value problem is self-adjoint, $\mathcal{L} = \mathcal{L}^*$.

6. Find the solvability condition that must be satisfied in order for the following boundary value problems to have a solution

$$
\begin{aligned}
&\text{(i)} \quad Lu = u'' = f(x), \quad B_1 u = u(0) - u(1) = c_1, \quad B_2 u = u'(0) - u'(1) = c_2;\\
&\text{(ii)} \quad Lu = u'' = f(x), \quad B_1 u = u(0) - u(1) = c_1, \quad B_2 u = u'(1) = c_2.
\end{aligned}
$$

Note that in (ii) the problem is not self-adjoint.