## Math 571 - Functional Analysis I - Fall 2017

### Homework 4

Due: **Friday, September 29, 2017**

1. (# 4, Section 2.1) Which is the following subsets of $\mathbb{R}^3$ constitute a subspace of $\mathbb{R}^3$? [Here, $x = (\xi_1, \xi_2, \xi_3)$.]

   (a) All $x$ with $\xi_1 = \xi_2$ and $\xi_3 = 0$.

   (b) All $x$ with $\xi_1 = \xi_2 + 1$.

   (c) All $x$ with positive $\xi_1$, $\xi_2$, $\xi_3$.

   (d) All $x$ with $\xi_1 - \xi_2 + \xi_3 = k = const.$

2. (# 5, Section 2.1) Show that $\{x_1, \ldots, x_n\}$, where $x_j(t) = t^j$, is a linearly independent set in the space $C[a, b]$.

3. (# 6, Section 2.1) Show that in an $n$-dimensional vector space $X$, the representation of any $x$ as linear combination of given basis vectors $e_1, \ldots, e_n$ is unique.

4. (# 10, Section 2.1) If $Y$ and $Z$ are subspaces of a vector space $X$, show that $Y \cap Z$ is a subspace of $X$, but $Y \cup Z$ need not be one. Give examples.

5. (# 4, Section 2.2) Show that we may replace (N2) by

   $$||x|| = 0 \quad \Rightarrow \quad x = 0$$

   without altering the concept of a norm. Show that nonnegativity of a norm also follows from (N3) and (N4).

6. (# 5, Section 2.2) Show that

   $$||x|| = \left( \sum_{j=1}^{n} |\xi_j|^2 \right)^{1/2} = \sqrt{|\xi_1|^2 + \ldots + |\xi_n|^2}$$

   in example 2.2-2 about Euclidean and unitary spaces defines a norm.

7. (# 6, Section 2.2) Let $X$ be the vector space of all ordered pairs $x = (\xi_1, \xi_2)$, $y = (\eta_1, \eta_2)$, ... of real numbers. Show that norms on $X$ are defined by

   $$||x||_1 = |\xi_1| + |\xi_2|$$

   $$||x||_2 = (\xi_1^2 + \xi_2^2)^{1/2}$$

   $$||x||_\infty = \max\{|\xi_1|, |\xi_2|\}.$$