Math 539 - Theory of Ordinary Differential Equations - Fall 2017

Homework 5

Due: **Friday, October 20, 2017**

1. Show that the boundary value problem

$$Lu = u'' + 2u' + u = f(x) \qquad x \in (0,1)$$
$$B_1 u = u(0) + mu'(0) = c_1 \qquad B_2 u = u(1) = c_2,$$

has a Green's function provided the parameter $m \neq \frac{1}{2}$. Find the Green's function when it exists. Why would you expect that your expression for $G(x,\xi)$ may not be symmetric in x and ξ?

2. Use a Green's function to solve the problem

$$\frac{d^2u}{dx^2} - \alpha^2 u = f(x) \qquad x \in (-\infty, \infty)$$
$$\text{where} \quad u \to 0 \quad \text{as} \quad x \to \pm\infty.$$

Assume that $\alpha > 0$ and f is sufficiently well-behaved as $x \to \pm\infty$ for a solution to exist.

3. When $\epsilon > 0$, find the Green's function $G(x,\xi)$ for the boundary value problem

$$Lu = \frac{d^2u}{dx^2} = f(x) \qquad x \in (0,1)$$
$$B_1 u = \frac{du}{dx}(0) = c_1 \qquad B_2 u = \epsilon u(1) + \frac{du}{dx}(1) = c_2.$$

(a) Use $G(x,\xi)$ to write down the solution $u_\epsilon(x)$ of the boundary value problem.
(b) Rearrange the solution as a two term expansion in ϵ, i.e., $u_\epsilon(x) = \frac{u_s}{\epsilon} + u_{r\epsilon}(x)$, where u_s is independent of ϵ. The first term in the expansion becomes unbounded as $\epsilon \to 0$ while the second term remains bounded.

As $\epsilon \to 0$ there is a necessary and sufficient condition on the data which ensures that the solution $u_\epsilon(x)$ remains bounded in the limit. It is just that $u_s = 0$.

(c) When $\epsilon = 0$ show that the homogeneous problem has a non-zero solution. Find the solvability condition given by the Fredholm alternative for the existence of a solution of the boundary value problem in this case.
(d) When $\epsilon = 0$, construct the modified Green's function $G_M(x,\xi)$ and use this to write down the (non-unique) solution $u_0(x)$ of the boundary value problem. Read the example done in-class to see that the modified Green's function here is *minus* that of the in-class example.
(e) Rearrange the solution $u_0(x)$, by using the solvability condition, to show

that it is the same (up to its non-uniqueness) as the regular part of the solution $u_{r\epsilon}(x)$ found in part (b).

This is the simplest example of its type but its features are typical. It should help to convince you (i) why a solvability condition is needed, and (ii) that the theoretical machinery of modified Green's functions 'works' to give the right answer.

4. Find the modified Green's function for the case $m = \frac{1}{2}$ of question 1.