MATH 326: Homework 6
SPRING 2013

1. Consider the polyhedral set P from HW # 5 (Problem 3) defined by the linear inequalities:

$$\begin{aligned} 3x_1 + x_2 &\geq 11 \\ x_1 + x_2 &\geq 5 \\ x_1 &\geq 3 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

   (a) Compute the equations and inequalities defining the set D of normalized directions (i.e. $e^T d = 1$).

   (b) Use D to identify the extreme directions of P. [Hint: Draw D as we did in class.]

   (c) Using the feasible region of D identify one non-extreme direction of P.

2. Consider the triangle (bounded polyhedral set) with extreme points $(-1, 0)$, $(1, 0)$ and $(0, 1)$. Show that the point $(0, 1/2)$ can be expressed as a convex combination of the extreme points. That is, find λ_1, λ_2 and λ_3 so that:

$$\lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$\lambda_1, \lambda_2, \lambda_3 \in [0, 1]$$

3. Let $X = \{x \in \mathbb{R}^n : \ \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0}\}$ and suppose that $\boldsymbol{d}_1, \dots, \boldsymbol{d}_l$ are the extreme directions of X (assuming it has any). Show that the problem:

$$\begin{cases} \min \ \boldsymbol{c}^T \boldsymbol{x} \\ \text{ s.t. } \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ \ \boldsymbol{x} \geq \boldsymbol{0} \end{cases}$$

   has a finite optimal solution if (and only if) $\boldsymbol{c}^T \boldsymbol{d}_j \geq 0$ for $k = 1, \dots, l$. [Hint: Modify the proof of Theorem about an optimal solution of Problem P from lecture 2/22/2013].

4. Consider the linear programming problem:

$$\begin{cases} \min \ z(x_1, x_2) = 2x_1 - x_2 \\ \text{s.t.} \ \ x_1 - x_2 \leq 1 \\ \qquad 2x_1 + x_2 \geq 6 \\ \qquad x_1, x_2 \geq 0 \end{cases}$$

We've shown in class that the extreme directions of the polyhedral feasible region are: $\boldsymbol{d}_1 = [0, 1]$ and $\boldsymbol{d}_2 = [1/2, 1/2]$. Use this fact and the statement from Problem 3 to show that this problem is unbounded. [Hint: $c^T = [2, -1]$. Show that $c^T \boldsymbol{d}_j < 0$ for either $j = 1$ or $j = 2$.]