**Math 432 - Numerical Linear Algebra - Fall 2013**

**Homework 6**

Assigned: Saturday, October 5, 2013

Due: **Friday, October 11, 2013**

- Include a cover page and a problem sheet.
- Include all of your scripts and output results.
- Place a comment at the top of each function or script that you submit which includes the name of the function or script

1. (a) Prove that the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

does not have an $LU$ decomposition.

(b) Does the system

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

have a unique solution for all $a, b \in \mathbb{R}$? (Why?)

(c) How can you modify the system in part (b) so that $LU$ decomposition applies?

2. Let $A = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix}$, $b = \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix}$, $x = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $x_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 0.9911 \\ -0.4870 \end{pmatrix}$.

(a) Show that $x$ is the exact solution of $Ax = b$.

(b) Think of $x_1$, $x_2$ as approximations to the exact solution $x$. Compute the errors $e_1$, $e_2$ and residuals $r_1$, $r_2$, corresponding to $x_1$, $x_2$ (see the definition of a residual below).

(c) Find $||A||$ , $||A^{-1}||$, Cond$(A)$.

(d) Let $A$ be an invertible matrix, $x$ be the exact solution of $Ax = b$, $\tilde{x}$ an approximate solution, $e = x - \tilde{x}$ an error and $r = b - A\tilde{x}$ the residual. Then (see also the lecture from 9/18/2013)

$$\frac{||e||}{||x||} \leq \mathrm{Cond}(A)\frac{||r||}{||b||}$$

Show that this result holds for the approximate solutions $x_1$, $x_2$ given above.

3. Derive the following result.

$$\left.\begin{aligned} Ax &= b \\ \tilde{A}\tilde{x} &= b \end{aligned}\right\} \quad \Rightarrow \quad \frac{||x - \tilde{x}||}{||\tilde{x}||} \leq \mathrm{Cond}(A)\frac{||A - \tilde{A}||}{||A||}$$

Note: This result says that in solving a linear system $Ax = b$, the condition number of the matrix controls the relative error in the solution due to perturbations in the matrix.

4. Let $A$ be a $3 \times 3$ matrix. Suppose we apply $LU$ factorization with partial pivoting and obtain $M_2 P_2 M_1 P_1 A = U$, where $U$ is upper triangular and

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_{32} & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

   (a) Compute $\tilde{M}_1 = P_2 M_1 P_2$.

   (b) Show that $P_2 M_1 = \tilde{M}_1 P_2$. Note that this implies $M_2 \tilde{M}_1 P_2 P_1 A = U$.

   (c) Compute $P = P_2 P_1$ and $L = \tilde{M}_1^{-1} M_2^{-1}$

   (d) Show that $PA = LU$.

   (e) Find $P$, $L$, $U$ such that $PA = LU$ and use the factorization to solve $Ax = b$, where

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

5. **(Partial Pivoting: MATLAB program)**

   Write a program to find the $LU$ decomposition of a given $n \times n$ matrix $A$ using **partial pivoting**. The program should return the updated matrix $A$ and the pivot vector $p$. In MATLAB, name your file mylu.m, the first few lines of which should be as follows:

```
function [a,p]=mylu(a)
%
[n n]=size(a); p=(1:n)'; (your code here!)
```

   The code above sets $n$ equal to the dimension of the matrix and initializes the pivot vector $p$. Make sure to store the multipliers $m_{ij}$ in the proper matrix entries. You should experiment with a few small matrices to make sure your code is correct. Check if matrices resulting in $LU$ decomposition satisfy $PA = LU$. As a test of your code, in MATLAB execute the statements

```
>>diary mylu.txt
>>format short e
>>type mylu.m
>>a=[2 2 -3;3 1 -2;6 8 1];
>>[a,p]=mylu(a)
>>diary off
```

   **Print and hand-in the text file containing your program.**