Math 539 - Theory of Ordinary Differential Equations - Fall 2017

<u>**Homework 6**</u>

Due: **Wednesday, November 1, 2017**

1. For the boundary value problem

$$u'' + 4\pi^2 u = f(x) \qquad x \in (0,1)$$
$$B_1 u = u(1) - u(0) = c_1 \qquad B_2 u = u'(1) - u'(0) = c_2$$

show that there is no Green's function and find the (two) solvability conditions that are necessary for a solution to exist.

2. (i) Rework example 2 from section 4.1 of the lecture notes with *in*homogeneous boundary conditions, i.e., find the solution of the boundary value problem

$$\frac{d^2u}{dx^2} = f(x) \qquad x \in (0,l)$$
$$B_1 u = u(0) = c_1 \qquad B_2 u = u(l) = c_2$$

as a series in terms of orthonormalized eigenfunctions of the homogeneous problem. First, recall from the lecture notes that the eigenvalue problem is

$$u_n'' + \lambda_n u_n = 0 \;\; x \in (0,l), \quad B_1 u_n = u_n(0) = 0, \quad B_2 u_n = u_n(l) = 0\,,$$

and the orthonormalized eigensystem is

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad u_n(x) = \sqrt{\frac{2}{l}}\sin\frac{n\pi x}{l}, \quad n = 1, 2, \dots\,,$$

which generates a Fourier sine series.

(ii) Decompose the solution into two parts $u = u_f + u_c$, where u_f is the response to f alone and u_c is the response to the boundary data (c_1, c_2) alone. Write down the boundary value problem for u_c and hence find the closed form expression for u_c in terms of solutions of the homogeneous differential equation. Use this and the solution of (i) to give the eigenfunction expansion of u_c, which is also its Fourier sine series.

3. Find the orthonormalized eigenvalues and eigenfunctions of the operator

$$Lu = -\frac{d^2u}{dx^2} \qquad x \in (0,l)$$
$$B_1 u = u(0) - u(l) \qquad B_2 u = u'(0) - u'(l).$$

Note that zero is also an eigenvalue.

Comments: The differential operator is of Sturm-Liouville type, but because the boundary conditions are periodic, and hence not separable, the eigenvalue problem is *not* a Sturm-Liouville eigenvalue problem, although it shares many of the features of one. One feature it does *not* share is that, while the eigenvalue zero has just one eigenfunction, each non-zero eigenvalue has *two* linearly independent eigenfunctions (i.e., each of the non-zero eigenvalues is 'degenerate' with eigenfunctions of multiplicity two). This is the eigensystem that forms the basis of the Fourier series of an l-periodic function (with both sines and cosines), where we often take $l = 2\pi$.

4. Form the inner product $\langle u, Lu\rangle$ to show that the operator

$$
\begin{aligned}
Lu &= -\frac{d^2u}{dx^2} \qquad x \in (0,1) \\
B_1u = u(0) &\qquad B_2u = u'(1) - \alpha u(1)
\end{aligned}
$$

   is positive definite when $\alpha < 0$.

   Find the eigenvalues and normalized eigenfunctions of the operator. Locate the eigenvalues by graphical solution or sketch, distinguishing (briefly) between the cases $\alpha < 1$, $\alpha = 1$, and $\alpha > 1$. Describe the position in the λ-plane of the *smallest* eigenvalue as α increases from $-\infty$ through $\alpha = 1$ to $\alpha = \infty$.

5. Find the eigenvalues and normalized eigenfunctions of the eigenvalue problem

$$
\begin{aligned}
Lu = -x(xu')' &= \lambda u \qquad x \in (1,2) \\
B_1u = u(1) = 0 &\qquad B_2u = u(2) = 0.
\end{aligned}
$$

   When carrying out the normalization, note that this is in fact a regular Sturm-Liouville eigenvalue problem, since the differential equation is of the form $(pu')' - qu + \lambda wu = 0$ with $p = x$, $q = 0$, and $w = \frac{1}{x}$, so that the inner product has weight $1/x$ and is $\langle f,\, g\rangle = \int_1^2 fg\,\frac{1}{x}dx$.