## Math 539 - Theory of Ordinary Differential Equations - Fall 2017

## Homework 7

## Due: Friday, November 10, 2017

1. Find the solution of the boundary value problem

$$
\begin{gathered}
(L-\mu) u=f(x) \quad x \in(0,1) \\
B_{1} u=u^{\prime}(0)=c_{1} \quad B_{2} u=u^{\prime}(1)=c_{2}
\end{gathered}
$$

where $L \equiv-d^{2} / d x^{2}$ by eigenfunction expansion. To do so, first show that the orthonormalized eigensystem of

$$
-u^{\prime \prime}=\lambda u \quad x \in(0,1), \quad B_{i} u=0 \quad i=1,2,
$$

is

$$
\lambda_{0}=0, u_{0}=1 ; \quad \lambda_{n}=(n \pi)^{2}, u_{n}(x)=\sqrt{2} \cos n \pi x, \quad n=1,2, \ldots
$$

Consider the case when the parameter $\mu$ is not an eigenvalue. Identify the part of the series solution that is the response to $f$ alone (i.e., with $c_{1}=c_{2}=0$ ) and use this to identify the eigenfunction expansion of the Green's function.
Suppose $\mu$ approaches an eigenvalue $\lambda_{n}$ for some value of $n$. Use the eigenfunction expansion above to find the condition that is necessary and sufficient for the solution to remain bounded as $\mu \rightarrow \lambda_{n}$, and give the eigenfunction expansion of the (non-unique) solution in this limit. Check that the condition for the solution to remain bounded is the same as the solvability condition provided by the Fredholm alternative.
2. (i) Find the solution of the boundary value problem

$$
\left(\left(1-x^{2}\right) u^{\prime}\right)^{\prime}=f(x) \quad x \in(-1,1) \quad u \text { bounded for } x \in[-1,1]
$$

in terms of eigenfunctions of the homogeneous problem. Verify that the eigenfunctions are the Legendre polynomials $u=P_{n}(x)$ with eigenvalues $\lambda_{n}=$ $n(n+1), n=0,1,2, \ldots$ What simple form does the expansion take when $f(x)=$ $a+b x+c x^{2}$; for this part you may use the result that $P_{n}(x)=\frac{1}{2^{n} n!} \frac{d^{n}}{d x^{n}}\left(x^{2}-1\right)^{n}$.
(ii) Given that the operator

$$
L u=-\left(\left(1-x^{2}\right) u^{\prime}\right)^{\prime} \quad x \in(-1,1) \quad u \text { bounded for } x \in[-1,1]
$$

has eigenvalues $\lambda_{n}=n(n+1)$ and eigenfunctions $u=P_{n}(x)$ for $n=0,1,2, \ldots$ where $P_{n}(x)$ is a Legendre polynomial, show that the operator

$$
L u=-\left(\left(1-x^{2}\right) u^{\prime}\right)^{\prime}+\frac{m^{2}}{1-x^{2}} u \quad x \in(-1,1) \quad u \text { bounded for } x \in[-1,1]
$$

has eigenvalues $\lambda_{n}=n(n+1)$ and eigenfunctions

$$
u=\left(1-x^{2}\right)^{m / 2} \frac{d^{m}}{d x^{m}} P_{n}(x)
$$

for $n=0,1,2, \ldots$ The latter are called the associated Legendre functions. Both types of function occur during separation of variables for Laplace's equation in spherical polars.
3. Let the operator $L$ be defined by

$$
L u \equiv-\frac{1}{x}\left(x u^{\prime}\right)^{\prime},
$$

with boundary conditions

$$
|u(0)|<\infty, \quad u(1)=0
$$

where primes denote $x$-derivatives.
(a) If the scalar product is defined by

$$
\langle u, v\rangle=\int_{0}^{1} x u(x) v(x) d x
$$

show that $L$ is self-adjoint and positive definite. What can be said about the sign of the eigenvalues of $L$ ?
(b) By considering the eigenvalue problem

$$
L u=\lambda u
$$

use the Rayleigh-Ritz theorem and an appropriate quadratic polynomial trial function to find an approximation to the first zero of $J_{0}(x)$, where $J_{0}$ is the Bessel function of the first kind and order zero which is bounded at the origin. Give a justification for your choice of trial function. You can quote the following facts about Bessel functions: the equation $\left(x u^{\prime}\right)^{\prime}+x u=0$ has independent solutions $J_{0}$ and $Y_{0}$. The solution $J_{0}$ is bounded at $x=0$ while $Y_{0}$ is not, with $J_{0}(0)=1$ and $J_{0}^{\prime}(0)=0$.
4. Use separation of variables to solve the Dirichlet problem for Laplace's equation on a disk:

$$
\nabla^{2} u=\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right)+\frac{1}{r^{2}} \frac{\partial^{2} u}{\partial \theta^{2}}=0 \quad r \in[0, a), \quad \theta \in(-\pi, \pi]
$$

$$
u \text { bounded, } \quad u(a, \theta)=f(\theta)
$$

Look for a separable solution of the form $u(r, \theta)=R(r) \Theta(\theta)$ and explain why $\Theta$ must be a periodic function of $\theta \in(-\pi, \pi]$. Show that the solution is a linear superposition of functions

$$
1, \text { and } r^{n} \cos n \theta, r^{n} \sin n \theta \quad n=1,2, \ldots,
$$

or equivalently 1 , and $r^{n} e^{ \pm i n \theta} n=1,2, \ldots$.
Show that you can write the solution in the form

$$
u=\frac{1}{2 \pi} \int_{-\pi}^{\pi} f(\phi) d \phi+\frac{1}{\pi} \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{n} \int_{-\pi}^{\pi} f(\phi) \cos n(\theta-\phi) d \phi
$$

5. Use the method of separation of variables to solve the problem

$$
\begin{gathered}
\nabla^{2} u=\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}}=0 \\
u(0, y)=0, u(l, y)=0, u(x, 0)=f(x), u(x, h)=0
\end{gathered}
$$

for Laplace's equation on the rectangle $0<x<l, 0<y<h$. Look for a separable solution of the form $u(x, y)=X(x) Y(y)$, and show that the solution is a superposition of functions

$$
\sin \frac{n \pi x}{l} \sinh \frac{n \pi}{l}(y-h) \quad n=1,2, \ldots .
$$

Hence show that the solution is

$$
\begin{aligned}
u & =\sum_{n=1}^{\infty} a_{n} \sin \frac{n \pi x}{l} \sinh \frac{n \pi}{l}(y-h) \\
a_{n} & =\frac{-2}{l \sinh \frac{n \pi h}{l}} \int_{0}^{l} f(x) \sin \frac{n \pi x}{l} d x
\end{aligned}
$$

How would you solve the problem if the boundary data were inhomogeneous on each of the four sides of the rectangle?

