Math 571 - Functional Analysis I - Fall 2017

Homework 7

Due: **Friday, October 20, 2017**

1. (# 2, Section 2.7) Let X and Y be normed spaces. Show that a linear operator $T : X \to Y$ is bounded if and only if T maps bounded sets in X into bounded sets in Y.

2. (# 3, Section 2.7) If $T \neq 0$ is a bounded linear operator, show that for any $x \in \mathcal{D}(T)$ such that $||x|| < 1$ we have strict inequality $||Tx|| < ||T||$.

3. (# 5, Section 2.7) Show that the operator $T : l^\infty \to l^\infty$ defined by $y = (\eta_j) = Tx$, $\eta_j = \xi_j / j$, $x = (\xi_j)$, is linear and bounded.

4. (# 6, Section 2.7) Show that the range $\mathcal{R}(T)$ of a bounded linear operator $T : X \to Y$ need not be closed in Y. Hint: Use T in Prob. # 5, Section 2.7.

5. (# 7, Section 2.7) Let T be a bounded linear operator from a normed space X onto a normed space Y. If there is a positive b such that

$$||Tx|| \geq b||x|| \quad \text{for all} \quad x \in X,$$

show that then $T^{-1} : Y \to X$ exists and is bounded.

6. (# 2, Section 2.8) Show that the functionals defined on $C[a, b]$ by

$$f_1(x) = \int_a^b x(t) y_0(t) dt, \quad y_0 \in C[a, b]$$

$$f_2(x) = \alpha x(a) + \beta x(b), \quad \alpha, \beta \text{ fixed}$$

are linear and bounded.

7. (# 3, Section 2.8) Find the norm of the linear functional f defined on $C[-1, 1]$ by

$$f(x) = \int_{-1}^0 x(t) dt - \int_0^1 x(t) dt.$$