## Math 539 - Theory of Ordinary Differential Equations - Fall 2017

## Homework 8

## Due: Monday, November 27, 2017

1. State whether or not the following boundary value problems have a unique solution, and when there is not a unique solution find the condition that must be satisfied in order for a solution to exist. Give an eigenfunction expansion of the (general) solution, and identify the eigenfunction expansion of the Green's or modified Green's function.
(i) $\quad u^{\prime \prime}+u=f(x), \quad u(0)=u(2 \pi), \quad u^{\prime}(0)=u^{\prime}(2 \pi)$

Use eigenfunctions of $L u=-u^{\prime \prime}-u$ with the same boundary conditions.
(ii) $u^{\prime \prime}=\sin x, \quad u^{\prime}(0)=\alpha, \quad u(\pi)=\beta$

Use eigenfunctions of $L u=-u^{\prime \prime}$ with homogeneous boundary conditions
2. Linear oscillations of a stretched string, clamped at both ends, with nonuniform density are governed by solutions of the boundary value problem

$$
\begin{gathered}
u^{\prime \prime}+\lambda(1+x) u=0 \quad x \in(0,1) \\
u(0)=0 \quad u(1)=0 .
\end{gathered}
$$

Here $\lambda(1+x)=\rho(\omega l)^{2} / T$ where the string has length $l$, variable line density $\rho(x)$, tension $T$, and executes harmonic oscillations with angular frequency $\omega$ and displacement $y(x, t)=\operatorname{Re}(u(x) \exp (i \omega t))$. The eigenfunctions for the boundary value problem can be related to error functions, but these are not easy to manipulate or interpret.
Apply the variational form of the Rayleigh-Ritz method (either by using the Rayleigh quotient $\rho(u)$ directly or by following page 122 of the notes) with the trial functions

$$
\psi_{i}(x)=x^{i}(1-x) \quad i=1,2, \ldots
$$

which are linearly independent, satisfy the boundary conditions for $u$, and are complete. Find an approximation to the first eigenpair by using the trial function $\psi_{1}$ alone, and then find an approximation to the first two eigenpairs by using a linear combination of $\psi_{1}$ and $\psi_{2}$.
Since the string is heavier at one end, the first eigenfunction is not symmetric about its midpoint $x=1 / 2$. Without solving to find the asymmetry of the first eigenfunction (which you can find from your second approximation to it) sketch the first eigenfunction by looking at the differential equation for $u$, indicating the light and heavy string ends. Write down the estimated ratio of the frequencies $\omega$ of the first two eigenvalues. (Your approximations should give $\hat{\lambda}_{1}^{(1)}=20 / 3=$ 6.667, then $\hat{\lambda}_{1}^{(2)}=6.634, \hat{\lambda}_{2}^{(2)}=28.592$ etc., This compares with an exact value of $\lambda_{1}=6.417$.)
3. Let

$$
\begin{aligned}
& L u=-\frac{d^{2}}{d x^{2}} \quad x \in(0, L) \\
& u(0)=0, \quad \frac{d u}{d x}(L)=0
\end{aligned}
$$

Find the Green's function $G(x, \xi, \mu)$ for the operator $L-\mu$. Use the Green's function and the result

$$
\delta(x-\xi)=-\frac{1}{2 \pi i} \int_{C_{\infty}} G(x, \xi, \mu) d \mu
$$

to find the spectral decomposition of $\delta(x-\xi)$ and define the corresponding transform pair. Here $C_{\infty}$ is a circle in the complex $\mu$-plane with center at the origin and radius $R$ in the limit $R \rightarrow \infty$.

