Math 539 - Theory of Ordinary Differential Equations - Fall 2017

<u>**Homework 9**</u>
Due: **Monday, December 4, 2017**

1. Consider the following two examples for the differential equation

$$Lu = -(p(x)u')' + q(x)u - \mu w(x)u = 0\,, \qquad x \in (a, b),$$

where μ is a parameter.

(a) The Legendre equation is $-((1 - x^2)u')' - \mu u = 0$ for $x \in (-1, 1)$ where the weight function is $w(x) = 1$. Show that both end-points, $x = -1$ and $x = 1$, are singular and explain why. Set the parameter $\mu = 0$ and show that two linearly independent homogeneous solutions are $u_1(x) = 1$ and $u_2(x) = \ln(1 + x) - \ln(1 - x)$. Consider the integrals $\int_{-1}^{l} u_i^2\,dx$ and $\int_{l}^{1} u_i^2\,dx$ for $i = 1, 2$ and $-1 < l < 1$ to show that both of the end-points are in the limit-circle case.

(b) Bessel's equation of order zero with parameter μ is $-(xu')' - \mu xu = 0$ for $x \in (0, b)$, where $0 < b < \infty$ and the weight function is $w(x) = x$. Show that for the end-points, $x = 0$ is singular and $x = b$ is regular and explain why. Set $\mu = 0$ and show that two linearly independent homogeneous solutions are $u_1(x) = 1$ and $u_2 = \ln x$. Consider the integral $\int_0^b u_i^2\,x\,dx$ for $i = 1, 2$ to show that the end-point $x = 0$ is in the limit circle case.

2. Let

$$Lu = -\frac{d^2}{dx^2} \qquad x \in (0, \infty)$$
$$\frac{du}{dx}(0) = 0, \qquad \int_0^\infty u^2(x)\,dx < \infty.$$

Find the Green's function $G(x, \xi, \mu)$ for the operator $L - \mu$. Use the Green's function and the result

$$\delta(x - \xi) = -\frac{1}{2\pi i}\int_{C_\infty} G(x, \xi, \mu)\,d\mu$$

to find the spectral decomposition of $\delta(x - \xi)$ and define the corresponding transform pair. Here C_∞ is a circle in the complex μ-plane with center at the origin and radius R in the limit $R \to \infty$.

3. Construct the Green's function $G(x, \xi; \mu)$ for the boundary value problem

$$-\frac{d^2G}{dx^2} - \mu G = \delta(x - \xi) \qquad G \in L^2(-\infty, \infty)$$

where μ is a complex parameter with $arg(\mu) \in [0, 2\pi)$. Show that both singular end points, $x = -\infty$ and $x = \infty$, are in the limit-point case and

$$G(x, \xi; \mu) = \frac{i}{2\sqrt{\mu}} e^{i\sqrt{\mu}|x-\xi|} \qquad -\infty < x, \xi < \infty$$

where $\mathrm{Im}(\sqrt{\mu}) \geq 0$.

Note that $G(x, \xi; \mu)$ has no poles in the complex μ-plane but has a branch cut along the positive real axis, and show that

$$\begin{array}{ll} \text{as } \mu \text{ approaches the cut from above} & G \to \dfrac{i}{2|\mu|^{1/2}} e^{i|\mu|^{1/2}|x-\xi|}, \\ \text{and as } \mu \text{ approaches the cut from below} & G \to \dfrac{-i}{2|\mu|^{1/2}} e^{-i|\mu|^{1/2}|x-\xi|}, \end{array}$$

where $|\mu|^{1/2}$ is real and positive.

Integrate $G(x, \xi; \mu)$ around a closed contour in the μ-plane which consists of a large circular arc C_∞ with center at the origin that does not cross the cut and a section C_B which goes from ∞ to zero below the cut and then from 0 to ∞ above the cut. Since $G(x, \xi; \mu)$ has no poles inside the closed contour, the integral around it is zero, and the theory tells us that

$$\frac{1}{2\pi i} \int_{C_\infty} G(x, \xi; \mu)\, d\mu = -\delta(x - \xi).$$

Hence show that

$$\delta(x - \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik|x-\xi|}\, dk.$$

Now multiply this equation by a function $f(\xi)$ and integrate with respect to ξ from $-\infty$ to ∞, to derive the Fourier transform pair

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} F(k)\, dk \\ F(k) &= \int_{-\infty}^{\infty} e^{ik\xi} f(\xi)\, d\xi \end{aligned}$$

(The function f needs to be piecewise C^1 and $L^2(-\infty, \infty)$ for this to work.)