Math 571 - Functional Analysis I - Fall 2017

Homework 9

Due: **Friday, November 3, 2017**

1. (# 9, Section 2.10) Show that a linear functional f on a vector space X is uniquely determined by its values on a Hamel basis for X.

2. (# 2, Section 3.1) (**Pythagorian theorem**) If $x \perp y$ in an inner product space X, show that (Fig. 1)
$$||x+y||^2 = ||x||^2 + ||y||^2.$$
Extend the formula to m mutually orthogonal vectors.

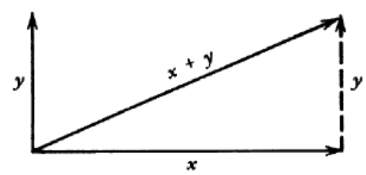

Figure 1: Illustration of the Pythagorean theorem in the plane

3. (# 3, Section 3.1) If X in problem 2 is real, show that, conversely, the given relation implies that $x \perp y$. Show that this may not hold if X is complex. Give examples.

4. (# 4, Section 3.1) If an inner product space X is real, show that the condition $||x|| = ||y||$ implies $\langle x+y, x-y\rangle = 0$. What does it mean geometrically if $X = \mathbb{R}^2$? What does the condition imply if X is complex?

5. (# 6, Section 3.1) Let $x \neq 0$, $y \neq 0$. (a) If $x \perp y$, show that $\{x, y\}$ is a linearly independent set. (b) Extend the result to mutually orthogonal vectors $x_1, \ldots, x_m$.

6. (# 4, Section 3.2) Show that $y \perp x_n$ and $x_n \to x$ together imply $x \perp y$.

7. (# 7, Section 3.2) Show that in an inner product space, $x \perp y$ if and only if we have $||x + \alpha y|| = ||x - \alpha y||$ for all scalars α. (See Fig. 2.)

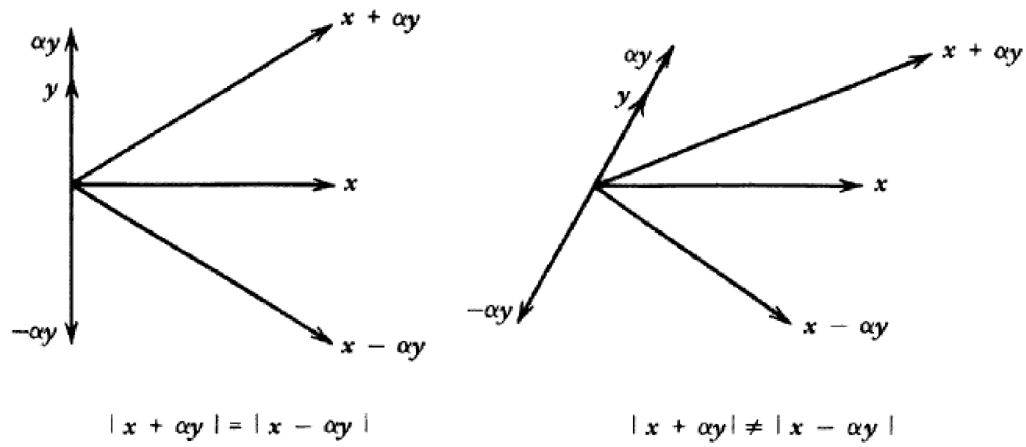

Figure 2: Illustration of Problem 7 in the Euclidean plane