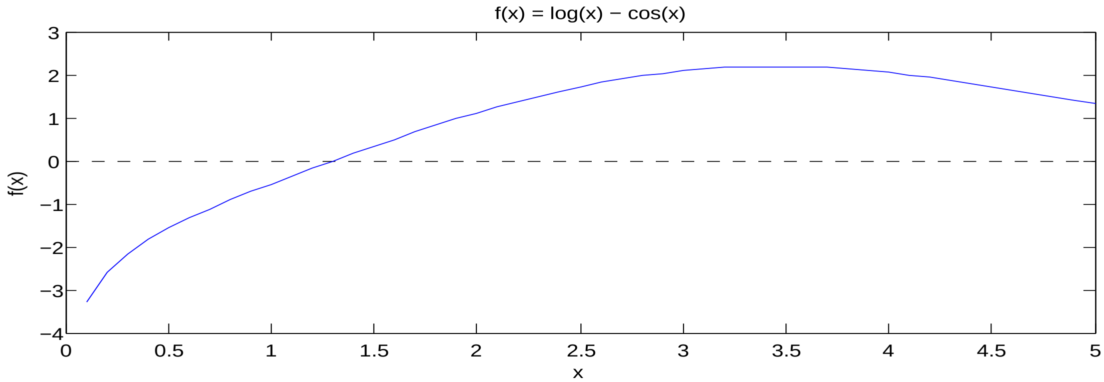
Math 471: Comparison of Rootfinding Methods

by Joel A. Tropp

We will apply four different numerical methods to approximate the (unique) root of $f(x) = \log(x) - \cos(x)$. There is no analytic formula for the root, but it lies between 1 and 1.5 by inspection:



For each method, we perform ten iterations, starting

- the Bisection and False Position Methods with interval $[a, b] = [1, 1.5]$,
- the Secant Method with $p_0 = 1$ and $p_1 = 1.5$, and
- Newton's Method with $p_0 = 1$.

Here are the results in numerical and graphical form.

Table 1: Approximation p_n versus iteration number n

n	Bisection	False Position	Secant	Newton
0	1.25000000000000	1.30873351472211	1.00000000000000	1.00000000000000
1	1.37500000000000	1.30313075185517	1.50000000000000	1.29340799302602
2	1.31250000000000	1.30296882214754	1.30873351472211	1.30295547292670
3	1.28125000000000	1.30296414059542	1.30285171617674	1.30296400120920
4	1.29687500000000	1.30296400524565	1.30296406194252	1.30296400121601
5	1.30468750000000	1.30296400133251	1.30296400121665	1.30296400121601
6	1.30078125000000	1.30296400121938	1.30296400121601	1.30296400121601
7	1.30273437500000	1.30296400121611	NaN	1.30296400121601
8	1.30371093750000	1.30296400121602	NaN	1.30296400121601
9	1.30322265625000	1.30296400121601	NaN	1.30296400121601

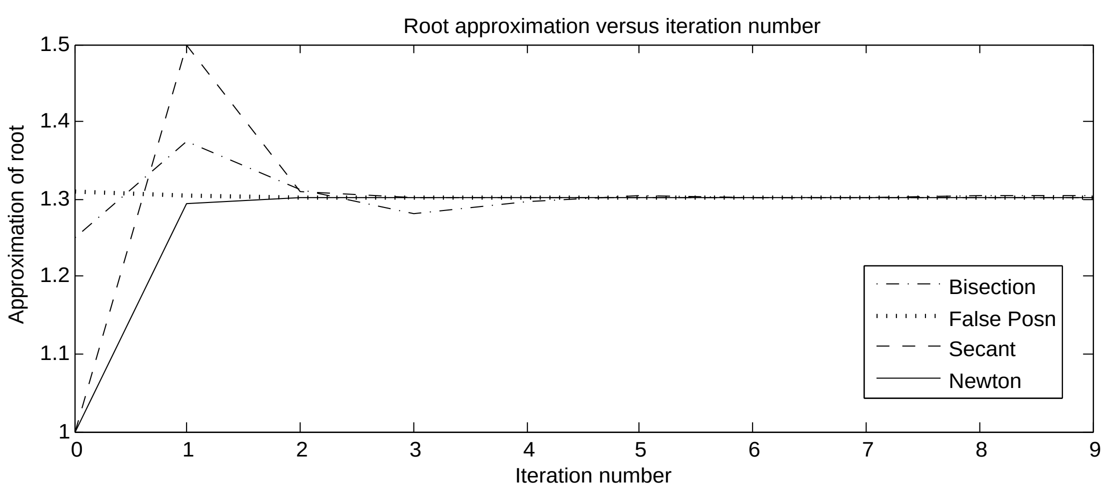
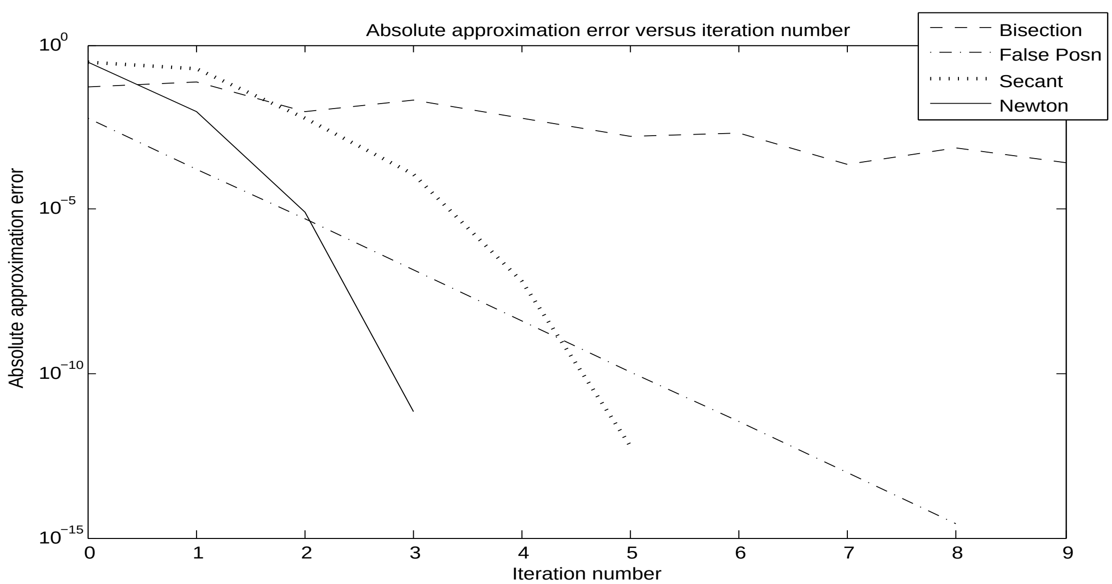


Table 2: Absolute error $|p_n - p|$ versus iteration number n

n	Bisection	False Position	Secant	Newton
0	0.05296400121601	0.00576951350610	0.30296400121601	0.30296400121601
1	0.07203599878399	0.00016675063916	0.19703599878399	0.00955600818999
2	0.00953599878399	0.00000482093153	0.00576951350610	0.00000852828932
3	0.02171400121601	0.00000013937940	0.00011228503928	0.00000000000681
4	0.00608900121601	0.00000000402964	0.00000006072651	0
5	0.00172349878399	0.00000000011650	0.00000000000064	0
6	0.00218275121601	0.00000000000337	0	0
7	0.00022962621601	0.00000000000010		0
8	0.00074693628399	0.00000000000000		0
9	0.00025865503399	0		0

Note: Zero entries indicate approximations that are exact to machine precision

Observe that the Bisection and False Position Methods add a constant number of correct digits in each iteration. This behavior corresponds with a linear order of convergence. On the other hand, Newton's Method *doubles* the number of correct digits at each iteration, which is the hallmark of a quadratically convergent method. The Secant Method improves the number of correct digits by a factor of $\phi = (\sqrt{5}+1)/2 \approx 1.6$. A semi-log plot makes these facts transparent:



In particular, the trend for the Method of False Position is almost perfectly *linear*, while the trend for Newton's Method is almost precisely a *quadratic* curve.

Finally, we estimate the asymptotic error constants by computing the appropriate ratio of the last two nontrivial approximation errors in each column.

Table 3: Approximate asymptotic error constants

Bisection:	$\lvert e_9\rvert / \lvert e_8\rvert$	= 0.35
False Posn:	$\lvert e_7\rvert / \lvert e_6\rvert$	= 0.03
Secant:	$\lvert e_5\rvert / \lvert e_4\rvert^{\phi}$	= 0.30
Newton:	$\lvert e_3\rvert / \lvert e_2\rvert^2$	= 0.09