Ex Evaluate the line integral by two methods: (a) directly and (b) using Green's Thm.

#1

S17.4

$$\oint_C (x-y)\,dx + (x+y)\,dy$$

where C is the circle with center the origin and radius 2.

(a)

$$\int_C P(x,y)\,dx + Q(x,y)\,dy =$$

$$= \int_a^b P(x(t), y(t)) \cdot x'(t)\,dt + Q(x(t), y(t)) \cdot y'(t)\,dt$$

$C: \quad x = x(t), \quad y = y(t), \quad a \le t \le b$

$$dx = x'(t)\,dt = \frac{dx}{dt}\,dt$$

In our case, $C: \ x^2+y^2 = 2^2$

or

$$x(t) = 2\cos t, \quad y = 2\sin t, \qquad 0 \le t \le 2\pi$$

$$\therefore \oint_C \underbrace{(x-y)}_{P(x,y)}\,dx + \underbrace{(x+y)}_{Q(x,y)}\,dy = \int_0^{2\pi} (2\cos t - 2\sin t) \circledast$$

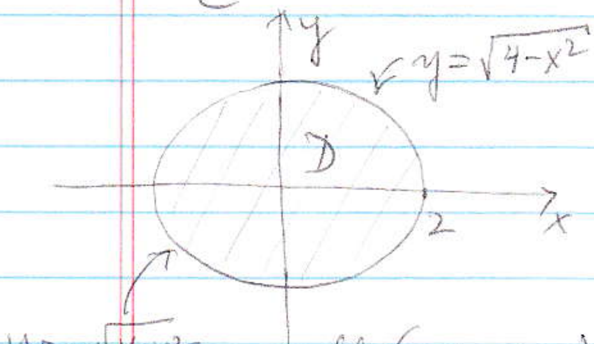
$$x'(t) = -2\sin t; \quad y'(t) = 2\cos t$$

$$\circledast \ (-2\sin t)\,dt + (2\cos t + 2\sin t) \cdot 2\cos t\,dt$$

$$= 4\int_0^{2\pi} \Big[ (\cancel{\cos t} - \sin t)(-\sin t) + (\cos t + \cancel{\sin t})\cos t \Big]\, dt$$

$$= 4\int_0^{2\pi} \underbrace{(\sin^2 t + \cos^2 t)}_{=1}\, dt = 4 \cdot 2\pi = 8\pi$$

(b) $\displaystyle\oint_C P\,dx + Q\,dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy \;=$



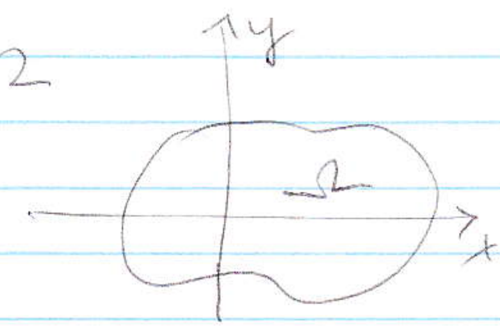

$P = x - y, \quad Q = x + y$

$\dfrac{\partial Q}{\partial x} = 1, \quad \dfrac{\partial P}{\partial y} = -1$

$$= \iint_D \big(1 - (-1)\big)\, dx\,dy = 2\iint_D dx\,dy = 2 \cdot \pi \cdot 2^2 = 8\pi$$

$\iint 1 \cdot dA = \text{area of } \Omega$

~~A~~ $\Omega$




$$\int_a^b 1 \cdot dx = b - a$$

$$\iint_D f(x,y)\,dx\,dy = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} f(x,y)\,dy\,dx$$

#6
S 17.3

Determine whether or not $\vec{F}$ is a conservative vector field. If it is, find a function $f$ such that $\vec{F} = \nabla f$.

$$\vec{F}(x,y) = (3x^2 - 2y^2)\vec{i} + (4xy+3)\vec{j}$$

Solution

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j} \qquad (1)$$

if $\vec{F}$ is conservative $\Rightarrow \exists f(x,y)$ such that

$$\nabla f = \vec{F}$$

$$\nabla f = f_x(x,y)\vec{i} + f_y(x,y)\vec{j} \qquad (2)$$

$(1),(2) \Rightarrow \quad P(x,y) = f_x, \quad Q(x,y) = f_y$

We know $f_{xy} = f_{yx}$ if $f$ has continuous 2nd order partial derivatives

$$f_{xy} = \frac{\partial P}{\partial y} \qquad f_{yx} = \frac{\partial Q}{\partial x} \Rightarrow \boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

$P(x,y) = 3x^2 - 2y^2 \qquad Q(x,y) = 4xy + 3$

$\frac{\partial P}{\partial y} = -4y \qquad \frac{\partial Q}{\partial x} = 4y$

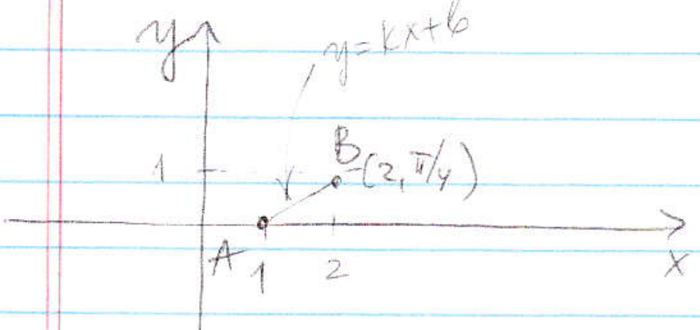
In general, $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ (unless $y=0$) $\Rightarrow$
$\vec{F}$ is not conservative

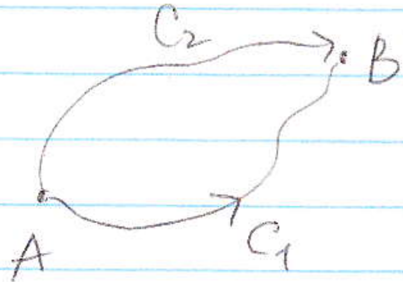
#19
S17.3

Show that the line integral is independent of path and evaluate the integral.

$$\int_C \underbrace{\tan y}_{P}\, dx + \underbrace{x \sec^2 y}_{Q}\, dy$$

C is any path from $(1,0)$ to $(2, \frac{\pi}{4})$




In general, $\int_{C_1} P(x,y)\,dx + Q(x,y)\,dy \neq \int_{C_2} P(x,y)\,dx + Q(x,y)\,dy$

Recall,

line $\int$ of vector field $\vec{F}$ along curve C

$$\int_C \vec{F}\, d\vec{r} = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

if $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$

C: $\vec{r} = \vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$

We know that $\int_C \vec{F}\, d\vec{r}$ is independent of path if $\vec{F}$ is conservative, i.e. $\exists f(x,y)$:

$$\nabla f = \vec{F}$$

$$\Rightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$P = \tan y, \quad Q = x \sec^2 y$$

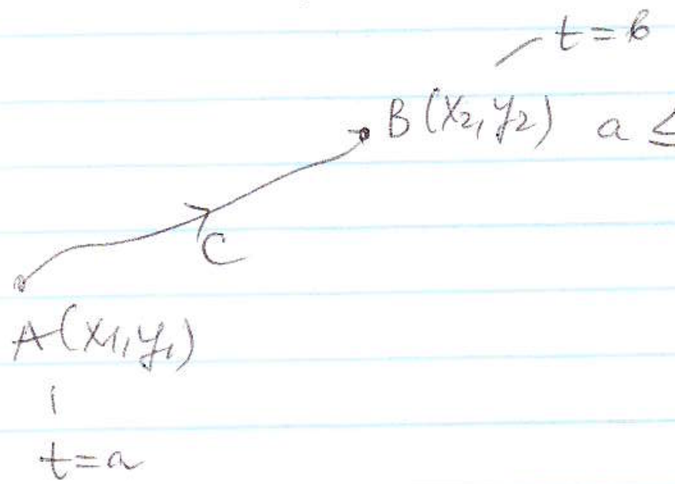
$$\frac{\partial P}{\partial y} = \sec^2 y \quad \frac{\partial Q}{\partial x} = \sec^2 y \quad \checkmark \quad \Rightarrow \vec{F} = P\vec{i} + Q\vec{j},$$

is conservative!

Recall

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = f(x_2, y_2) - f(x_1, y_1)$$

$$f = f(x,y)$$




$a \le t \le b$

or

$$\int_{x=1}^{2} \tan(kx+b)\,dx + x \cdot \sec^2(kx+b)\, k\, dx$$

Find potential $f$

$$\vec{F} = P\hat{\imath} + Q\hat{\jmath} = f_x\hat{\imath} + f_y\hat{\jmath}$$

$$P = f_x \qquad Q = f_y$$

$$f_x = P = \tan y \Rightarrow f(x,y) = x\tan y + g(y)$$

$$f_y = \underbrace{x\sec^2 y}_{} + g'(y) = Q = \underbrace{x\sec^2 y}_{}$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = K = \text{const}$$

$$\Rightarrow \boxed{f(x,y) = x\tan y + K}$$

$$\therefore \int_C \tan y\, dx + x\sec^2 y\, dy = f\left(2,\tfrac{\pi}{4}\right) - f(1,0)$$

$f\left(2,\tfrac{\pi}{4}\right)$: terminal pt of C    $f(1,0)$: initial pt of C

$$= \left(x\tan y + \cancel{K}\right)\Big|_{\substack{x=2\\ y=\frac{\pi}{4}}} - \left(x\tan y + \cancel{K}\right)\Big|_{\substack{x=1\\ y=0}} =$$

$$= 2\underbrace{\tan\tfrac{\pi}{4}}_{=1} - \cancel{1\cdot\tan 0}^{\,0} = \boxed{2}$$

Another review: this Sunday, May 8, at 4pm
TLC 028