Multistep Methods

.1 General 2-step method

$$\alpha_0 u_{n+1} + \alpha_1 u_n + \alpha_2 u_{n-1} = h\left(\beta_0 f(u_{n+1}) + \beta_1 f(u_n) + \beta_2 f(u_{n-1})\right)$$

Note

1. We assume $\alpha_0 \neq 0$.
2. If $\beta_0 = 0$, the scheme is *explicit*.

   If $\beta_0 \neq 0$, the scheme is *implicit*.

.2 Adams-Bashforth Methods

$$u_{n+1} = u_n + \frac{h}{2}\left(3f(u_n) - f(u_{n-1})\right)$$

explicit, 2nd order accurate, 1 function evaluation per step.

.3 Adams-Moulton

$$u_{n+1} = u_n + \frac{h}{12}\left(5f(u_{n+1}) + 8f(u_n) - f(u_{n-1})\right)$$

implicit, 3rd order accurate

Note

1. There are k-step AB and AM methods.

   Popular AB methods:

$$u_{n+1} = u_n + \frac{h}{12}\left(23f(u_n) - 16f(u_{n-1}) + 5f(u_{n-2})\right) : \quad \text{AB 3rd order}$$

$$u_{n+1} = u_n + \frac{h}{24}\left(55f(u_n) - 59f(u_{n-1}) + 37f(u_{n-2}) - 9f(u_{n-3})\right) :$$

AB 4th order

Other higher order AM methods:

$$u_{n+1} = u_n + \frac{h}{24}\left(9f(u_{n+1}) + 19f(u_n) - 5f(u_{n-1}) + f(u_{n-2})\right):$$

AM 4th order

$$u_{n+1} = u_n + \frac{h}{720}\left(251f(u_{n+1}) + 646f(u_n) - 264f(u_{n-1}) - 106f(u_{n-2}) - 19f(u_{n-3})\right): \quad \text{AM 5th order}$$

2. A popular predictor corrector method uses AB as predictor and AM as corrector.

.4 Leap-Frog

$$\frac{u_{n+1} - u_{n-1}}{2h} = f(u_n)$$

$$u_{n+1} = u_{n-1} + 2hf(u_n)$$

explicit, 2nd order accurate, 1 function evaluation per step

.5 BDF: Backward Differentiation Formula — Gear's Method

$$\frac{3}{2}u_{n+1} - 2u_n + \frac{1}{2}u_{n-1} = hf(u_n)$$

explicit, 2nd order accurate, 1 function evaluation per step

.6 Software

adaptive time step: ODE45 (Runge-Kutta (4,5) method, non-stiff DEs), ODE23 (Runge-Kutta (2,3) method, can be used for moderate stiffness, very often more efficient at crude tolerances than ODE45), ODE23TB (stiff DEs), ODE23S (stiff DEs), ODE23t (implements trapezoid rule using "free" interpolant, moderate stiffness), ODE23TB (implicit Runge-Kutta method: trapezoid rule step + BDF) in Matlab

adaptive order: ODE113 (variable order ABM predictor-corrector), ODE15S (variable order solver based on numerical differentiation formulas (NDFs), optionally uses BDFs - Gear's method, stiff DEs)