6. Write the second-order initial-value problems (11.3) and (11.4) as first-order systems, and derive the equations necessary to solve the systems using the fourth-order Runge-Kutta method for systems.

7. Let u represent the electrostatic potential between two concentric metal spheres of radii R_1 and R_2, with $R_1 < R_2$, such that the potential of the inner sphere is kept constant at V_1 volts and the potential of the outer sphere is 0 volts. The potential in the region between the two spheres is governed by Laplace's equation, which, in this particular application, reduces to

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} = 0, \quad R_1 \le r \le R_2, \quad u(R_1) = V_1, \quad u(R_2) = 0.$$

Suppose $R_1 = 2$ in., $R_2 = 4$ in., and $V_1 = 110$ volts.

   a. Approximate $u(3)$ using the Linear Shooting Algorithm.

   b. Compare the results of part (a) with the actual potential $u(3)$, where

$$u(r) = \frac{V_1 R_1}{r}\left(\frac{R_2 - r}{R_2 - R_1}\right).$$

8. Show that if y_2 is the solution to $y'' = p(x)y' + q(x)y$ and $y_2(a) = y_2(b) = 0$, then $y_2 \equiv 0$.

9. Consider the boundary-value problem

$$y'' + y = 0, \quad 0 \le x \le b, \quad y(0) = 0, \quad y(b) = B.$$

Find choices for b and B so that the boundary-value problem has

   a. No solution;

   b. Exactly one solution;

   c. Infinitely many solutions.

10. Attempt to apply Exercise 9 to the boundary-value problem

$$y'' - y = 0, \quad 0 \le x \le b, \quad y(0) = 0, \quad y(b) = B.$$

What happens? How do both problems relate to Corollary 11.2?

11.2 The Shooting Method for Nonlinear Problems

The shooting technique for the nonlinear second-order boundary-value problem

$$y'' = f(x, y, y'), \quad a \le x \le b, \quad y(a) = \alpha, \quad y(b) = \beta, \tag{11.6}$$

is similar to the linear case, except that the solution to a nonlinear problem cannot be expressed as a linear combination of the solutions to two initial-value problems. Instead, we need to use the solutions to a *sequence* of initial-value problems of the form

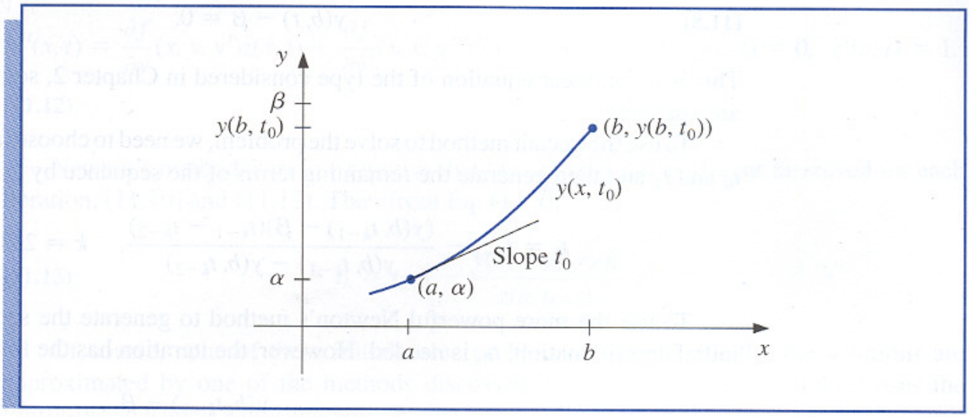
$$y'' = f(x, y, y'), \quad a \le x \le b, \quad y(a) = \alpha, \quad y'(a) = t, \tag{11.7}$$

involving a parameter t, to approximate the solution to the boundary-value problem. We do this by choosing the parameters $t = t_k$ so that

$$\lim_{k \to \infty} y(b, t_k) = y(b) = \beta,$$

where $y(x, t_k)$ denotes the solution to the initial-value problem (11.7) with $t = t_k$ and $y(x)$ denotes the solution to the boundary-value problem (11.6).

Figure 11.2

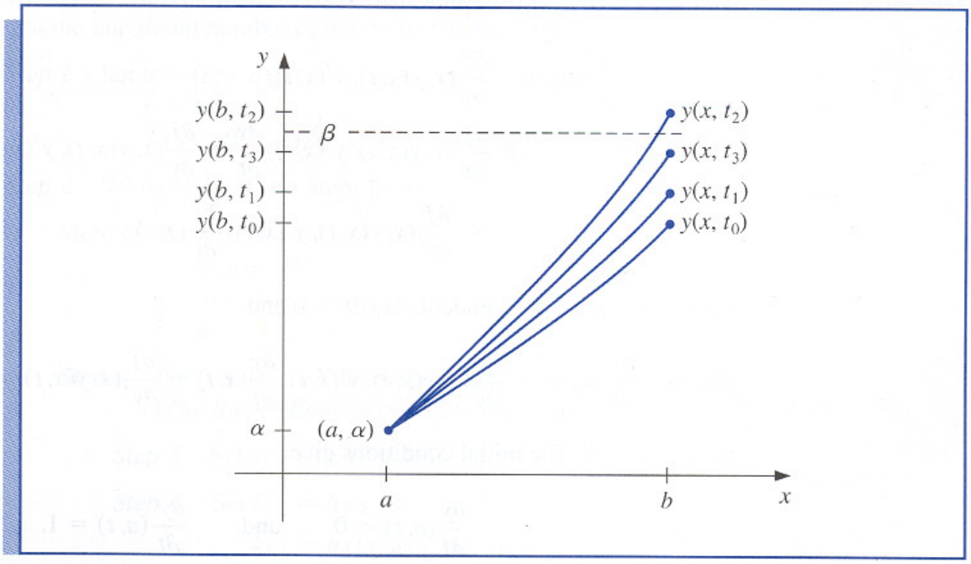


This technique is called a "shooting" method, by analogy to the procedure of firing objects at a stationary target. (See Figure 11.2.) We start with a parameter t_0 that determines the initial elevation at which the object is fired from the point (a, α) and along the curve described by the solution to the initial-value problem:

$$y'' = f(x, y, y'), \quad a \le x \le b, \quad y(a) = \alpha, \quad y'(a) = t_0.$$

If $y(b, t_0)$ is not sufficiently close to β, we correct our approximation by choosing elevations t_1, t_2, and so on, until $y(b, t_k)$ is sufficiently close to "hitting" β. (See Figure 11.3.)

Figure 11.3



To determine the parameters t_k, suppose a boundary-value problem of the form (11.6) satisfies the hypotheses of Theorem 11.1. If $y(x, t)$ denotes the solution to the initial-value problem (11.7), the problem is to determine t so that

(11.8) $$y(b,t) - \beta = 0.$$

This is a nonlinear equation of the type considered in Chapter 2, so a number of methods are available.

To use the Secant method to solve the problem, we need to choose initial approximations t_0 and t_1 and then generate the remaining terms of the sequence by

$$t_k = t_{k-1} - \frac{(y(b, t_{k-1}) - \beta)(t_{k-1} - t_{k-2})}{y(b, t_{k-1}) - y(b, t_{k-2})}, \quad k = 2, 3, \ldots.$$

To use the more powerful Newton's method to generate the sequence $\{t_k\}$, only one initial approximation, t_0, is needed. However, the iteration has the form

(11.9) $$t_k = t_{k-1} - \frac{y(b, t_{k-1}) - \beta}{\dfrac{dy}{dt}(b, t_{k-1})},$$

and requires the knowledge of $(dy/dt)(b, t_{k-1})$. This presents a difficulty since an explicit representation for $y(b, t)$ is not known; we know only the values $y(b, t_0), y(b, t_1), \ldots, y(b, t_{k-1})$.

Suppose we rewrite the initial-value problem (11.7), emphasizing that the solution depends on both x and t as

(11.10) $$y''(x,t) = f(x, y(x,t), y'(x,t)), \quad a \le x \le b, \quad y(a,t) = \alpha, \quad y'(a,t) = t.$$

We have retained the prime notation to indicate differentiation with respect to x. Since we need to determine $(dy/dt)(b, t)$ when $t = t_{k-1}$, we first take the partial derivative of (11.10) with respect to t. This implies that

$$\begin{aligned}
\frac{\partial y''}{\partial t}(x,t) &= \frac{\partial f}{\partial t}(x, y(x,t), y'(x,t)) \\
&= \frac{\partial f}{\partial x}(x, y(x,t), y'(x,t))\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}(x, y(x,t), y'(x,t))\frac{\partial y}{\partial t}(x,t) \\
&\quad + \frac{\partial f}{\partial y'}(x, y(x,t), y'(x,t))\frac{\partial y'}{\partial t}(x,t).
\end{aligned}$$

Since x and t are independent, $\partial x/\partial t = 0$ and

(11.11) $$\frac{\partial y''}{\partial t}(x,t) = \frac{\partial f}{\partial y}(x, y(x,t), y'(x,t))\frac{\partial y}{\partial t}(x,t) + \frac{\partial f}{\partial y'}(x, y(x,t), y'(x,t))\frac{\partial y'}{\partial t}(x,t)$$

for $a \le x \le b$. The initial conditions give

$$\frac{\partial y}{\partial t}(a,t) = 0 \quad \text{and} \quad \frac{\partial y'}{\partial t}(a,t) = 1.$$

If we simplify the notation by using $z(x,t)$ to denote $(\partial y/\partial t)(x,t)$ and assume that the order of differentiation of x and t can be reversed, (11.11) with the initial conditions becomes the initial-value problem

$$z''(x,t) = \frac{\partial f}{\partial y}(x,y,y')z(x,t) + \frac{\partial f}{\partial y'}(x,y,y')z'(x,t), \quad a \le x \le b, \quad z(a,t)=0, \quad z'(a,t)=1. \tag{11.12}$$

Newton's method therefore requires that two initial-value problems be solved for each iteration, (11.10) and (11.12). Then from Eq. (11.9),

$$t_k = t_{k-1} - \frac{y(b,t_{k-1})-\beta}{z(b,t_{k-1})}. \tag{11.13}$$

Of course, none of these initial-value problems is solved exactly; the solutions are approximated by one of the methods discussed in Chapter 5. Algorithm 11.2 uses the fourth-order Runge-Kutta method to approximate both solutions required by Newton's method. A similar procedure for the Secant method is considered in Exercise 4.

ALGORITHM **11.2**

Nonlinear Shooting

To approximate the solution of the nonlinear boundary-value problem

$$y'' = f(x,y,y'), \quad a \le x \le b, \quad y(a)=\alpha, \quad y(b)=\beta:$$

(Note: Equations (11.10) *and* (11.12) *are written as first-order systems and solved.)*

INPUT endpoints a, b; boundary conditions α, β; number of subintervals $N \ge 2$; tolerance TOL; maximum number of iterations M.

OUTPUT approximations $w_{1,i}$ to $y(x_i)$; $w_{2,i}$ to $y'(x_i)$ for each $i = 0, 1, \ldots, N$ or a message that the maximum number of iterations was exceeded.

Step 1 Set $h = (b-a)/N$;
$k = 1$;
$TK = (\beta - \alpha)/(b-a)$. *(Note: TK could also be input.)*

Step 2 While ($k \le M$) do Steps 3–10.

Step 3 Set $w_{1,0} = \alpha$;
$w_{2,0} = TK$;
$u_1 = 0$;
$u_2 = 1$.

Step 4 For $i = 1, \ldots, N$ do Steps 5 and 6.
(The Runge-Kutta method for systems is used in Steps 5 *and* 6.*)*

Step 5 Set $x = a + (i-1)h$.

Step 6 Set $k_{1,1} = hw_{2,i-1}$;
$k_{1,2} = hf(x, w_{1,i-1} w_{2,i-1})$;

$$k_{2,1} = h\left(w_{2,i-1} + \tfrac{1}{2}k_{1,2}\right);$$
$$k_{2,2} = hf\left(x + h/2, w_{1,i-1} + \tfrac{1}{2}k_{1,1}, w_{2,i-1} + \tfrac{1}{2}k_{1,2}\right);$$
$$k_{3,1} = h\left(w_{2,i-1} + \tfrac{1}{2}k_{2,2}\right);$$
$$k_{3,2} = hf\left(x + h/2, w_{1,i-1} + \tfrac{1}{2}k_{2,1}, w_{2,i-1} + \tfrac{1}{2}k_{2,2}\right);$$
$$k_{4,1} = h(w_{2,i-1} + k_{3,2});$$
$$k_{4,2} = hf(x + h, w_{1,i-1} + k_{3,1}, w_{2,i-1} + k_{3,2});$$
$$w_{1,i} = w_{1,i-1} + (k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})/6;$$
$$w_{2,i} = w_{2,i-1} + (k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2})/6;$$
$$k'_{1,1} = hu_2;$$
$$k'_{1,2} = h[f_y(x, w_{1,i-1}, w_{2,i-1})u_1 + f_{y'}(x, w_{1,i-1}, w_{2,i-1})u_2];$$
$$k'_{2,1} = h\left[u_2 + \tfrac{1}{2}k'_{1,2}\right];$$
$$k'_{2,2} = h\left[f_y(x + h/2, w_{1,i-1}, w_{2,i-1})\left(u_1 + \tfrac{1}{2}k'_{1,1}\right) + f_{y'}(x + h/2, w_{1,i-1}, w_{2,i-1})\left(u_2 + \tfrac{1}{2}k'_{1,2}\right)\right];$$
$$k'_{3,1} = h\left(u_2 + \tfrac{1}{2}k'_{2,2}\right);$$
$$k'_{3,2} = h\left[f_y(x + h/2, w_{1,i-1}, w_{2,i-1})\left(u_1 + \tfrac{1}{2}k'_{2,1}\right) + f_{y'}(x + h/2, w_{1,i-1}, w_{2,i-1})\left(u_2 + \tfrac{1}{2}k'_{2,2}\right)\right];$$
$$k'_{4,1} = h(u_2 + k'_{3,2});$$
$$k'_{4,2} = h\left[f_y(x + h, w_{1,i-1}, w_{2,i-1})\left(u_1 + k'_{3,1}\right) + f_{y'}(x + h, w_{1,i-1}, w_{2,i-1})\left(u_2 + k'_{3,2}\right)\right];$$
$$u_1 = u_1 + \tfrac{1}{6}[k'_{1,1} + 2k'_{2,1} + 2k'_{3,1} + k'_{4,1}];$$
$$u_2 = u_2 + \tfrac{1}{6}[k'_{1,2} + 2k'_{2,2} + 2k'_{3,2} + k'_{4,2}].$$

Step 7 If $|w_{1,N} - \beta| \le TOL$ then do Steps 8 and 9.

Step 8 For $i = 0, 1, \ldots, N$
set $x = a + ih$;
OUTPUT $(x, w_{1,i}, w_{2,i})$.

Step 9 (*Procedure is complete.*)
STOP.

Step 10 Set $TK = TK - \left(\dfrac{w_{1,N} - \beta}{u_1}\right)$; (*Newton's method is used to compute TK.*)
$k = k + 1$.

Step 11 OUTPUT ('Maximum number of iterations exceeded');
(*Procedure completed unsuccessfully.*)
STOP.

In Step 7, the best approximation to β we can expect for $w_{1,N}(t_k)$ is $O(h^n)$, if the approximation method selected for Step 6 gives $O(h^n)$ rate of convergence.

The value $t_0 = TK$ selected in Step 1 is the slope of the straight line through (a, α) and (b, β). If the problem satisfies the hypotheses of Theorem 11.1, any choice of t_0 will give convergence; but given a good choice of t_0, the convergence will improve and the procedure will work for many problems that do not satisfy these hypotheses.

EXAMPLE 1 Consider the boundary-value problem

$$y'' = \frac{1}{8}(32 + 2x^3 - yy'), \quad 1 \le x \le 3, \quad y(1) = 17, \quad y(3) = \frac{43}{3},$$

which has the exact solution $y(x) = x^2 + 16/x$.

Applying the Shooting method given in Algorithm 11.2 to this problem requires approximating solutions to the initial-value problems

$$y'' = \frac{1}{8}(32 + 2x^3 - yy'), \quad 1 \le x \le 3, \quad y(1) = 17, \quad y'(1) = t_k,$$

and

$$z'' = \frac{\partial f}{\partial y}z + \frac{\partial f}{\partial y'}z' = -\frac{1}{8}(y'z + yz'), \quad 1 \le x \le 3, \quad z(1) = 0, \quad z'(1) = 1,$$

at each step in the iteration.

Table 11.2

x_i	$w_{1,i}$	$y(x_i)$	$\lvert w_{1,i} - y(x_i)\rvert$
1.0	17.000000	17.000000	
1.1	15.755495	15.755455	4.06×10^{-5}
1.2	14.773389	14.773333	5.60×10^{-5}
1.3	13.997752	13.997692	5.94×10^{-5}
1.4	13.388629	13.388571	5.71×10^{-5}
1.5	12.916719	12.916667	5.23×10^{-5}
1.6	12.560046	12.560000	4.64×10^{-5}
1.7	12.301805	12.301765	4.02×10^{-5}
1.8	12.128923	12.128889	3.14×10^{-5}
1.9	12.031081	12.031053	2.84×10^{-5}
2.0	12.000023	12.000000	2.32×10^{-5}
2.1	12.029066	12.029048	1.84×10^{-5}
2.2	12.112741	12.112727	1.40×10^{-5}
2.3	12.246532	12.246522	1.01×10^{-5}
2.4	12.426673	12.426667	6.68×10^{-6}
2.5	12.650004	12.650000	3.61×10^{-6}
2.6	12.913847	12.913846	9.17×10^{-7}
2.7	13.215924	13.215926	1.43×10^{-6}
2.8	13.554282	13.554286	3.46×10^{-6}
2.9	13.927236	13.927241	5.21×10^{-6}
3.0	14.333327	14.333333	6.69×10^{-6}

If the stopping technique

$$|w_{1,N}(t_k) - y(3)| \leq 10^{-5}$$

is used, this problem requires four iterations and $t_4 = -14.000203$. The results obtained for this value of t are shown in Table 11.2. ■

Although Newton's method used with the shooting technique requires the solution of an additional initial-value problem, it will generally be faster than the Secant method. Both methods are only locally convergent, since they require good initial approximations. For a general discussion of the convergence of the shooting techniques for nonlinear problems, the reader is referred to the excellent book by Keller [K,H]. In that reference, more general boundary conditions are discussed. It is also noted that the shooting technique for nonlinear problems is sensitive to round-off errors, especially if the solution $y(x)$ and $z(x, t)$ are rapidly increasing functions on $[a, b]$.

EXERCISE SET 11.2

1. Use the Nonlinear Shooting Algorithm with $h = 0.5$ to approximate the solution to the boundary-value problem

$$y'' = -(y')^2 - y + \ln x, \quad \text{for} \quad 1 \leq x \leq 2, \quad \text{where} \quad y(1) = 0 \quad \text{and} \quad y(2) = \ln 2.$$

Compare your results to the actual solution $y(x) = \ln x$.

2. Use the Nonlinear Shooting Algorithm with $h = 0.25$ to approximate the solution to the boundary-value problem

$$y'' = 2y^3, \quad \text{for} \quad 1 \leq x \leq 2, \quad \text{where} \quad y(1) = \frac{1}{4} \quad \text{and} \quad y(2) = \frac{1}{5}.$$

Compare your results to the actual solution $y(x) = 1/(x + 3)$.

3. Use the Nonlinear Shooting method with $TOL = 10^{-4}$ to approximate the solution to the following boundary-value problems. The actual solution is given for comparison to your results.

   a. $y'' = y^3 - yy'$, $1 \leq x \leq 2$, $y(1) = \frac{1}{2}$, $y(2) = \frac{1}{3}$; use $h = 0.1$ and compare the results to $y(x) = (x + 1)^{-1}$.

   b. $y'' = 2y^3 - 6y - 2x^3$, $1 \leq x \leq 2$, $y(1) = 2$, $y(2) = \frac{5}{2}$; use $h = 0.1$ and compare the results to $y(x) = x + x^{-1}$.

   c. $y'' = y' + 2(y - \ln x)^3 - x^{-1}$, $1 \leq x \leq 2$, $y(1) = 1$, $y(2) = \frac{1}{2} + \ln 2$; use $h = 0.1$ and compare the results to $y(x) = x^{-1} + \ln x$.

   d. $y'' = [x^2(y')^2 - 9y^2 + 4x^6]/x^5$, $1 \leq x \leq 2$, $y(1) = 0$, $y(2) = \ln 256$; use $h = 0.05$ and compare the results to $y(x) = x^3 \ln x$.

4. Change Algorithm 11.2 to incorporate the Secant method instead of Newton's method. Use $t_0 = (\beta - \alpha)/(b - a)$ and $t_1 = t_0 + (\beta - y(b, t_0))/(b - a)$.

5. Repeat Exercise 3(a) and 3(c) using the Secant algorithm derived in Exercise 4, and compare the number of iterations required for the two methods.