1 Aligned-Rank Transform Methods

We have seen that rank-based methods such as the Kruskal-Wallis test for a completely randomized design are useful nonparametric alternatives to the analysis of variance (ANOVA). These methods were further advanced by Iman and Conover (1981), who discussed the concept of using standard parametric tests applied to ranked data. However, the rank-transform method can fail when applied to more complicated models, such as a factorial ANOVA model with interaction.

An alternative to the rank-transform method for multifactor experiments is the aligned-rank transform method. We will describe the aligned-rank transform method for a completely randomized factorial design with two factors and balanced data (replications within treatment combinations are all equal). The standard model for a completely randomized factorial design with two factors, including interaction, is:

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk},$$

where Y_{ijk} is the k^{th} replicate observation from level i of factor A and level j of factor B. With (typical) sum-to-zero restrictions on the parameters, the least-squares estimates are:

$$\widehat{\alpha}_i = \widehat{\mu}_{i.} - \widehat{\mu},\ \widehat{\beta}_j = \widehat{\mu}_{.j} - \widehat{\mu},\ \widehat{\gamma}_{ij} = \widehat{\mu}_{ij} - \widehat{\mu}_{i.} - \widehat{\mu}_{.j} + \widehat{\mu},$$

where

$$\widehat{\mu}_{i.} = \frac{1}{nb}\sum_j \sum_k Y_{ijk},\ \widehat{\mu}_{.j} = \frac{1}{na}\sum_i \sum_k Y_{ijk},\ \widehat{\mu} = \frac{1}{nab}\sum_i \sum_j \sum_k Y_{ijk}$$

The idea behind the aligned-rank transform method is to create a separate term for each effect in the model (main effects and interactions) by subtracting out all other effects. Then these separate terms (the **aligned effects**) are ranked and analyzed. For the interaction term and the main effects of factors A and B, the aligned effects are:

$$AB_{ijk} = Y_{ijk} - \widehat{\mu} - \widehat{\alpha}_i - \widehat{\beta}_j,\ A_{ijk} = Y_{ijk} - \widehat{\mu} - \widehat{\beta}_j - \widehat{\gamma}_{ij},\ \text{and } B_{ijk} = Y_{ijk} - \widehat{\mu} - \widehat{\alpha}_i - \widehat{\gamma}_{ij}.$$

Note that these aligned effects are always calculated using the estimates of μ, α, β, and γ from the full model above which includes all terms. Each of these three aligned effects is then ranked and analyzed via two-way ANOVA. Since the AB_{ijk} term has had both main effects removed, the only test of interest for it is the interaction effect, and similarly the only test of interest for A_{ijk} is the main effect of factor A, and the only test of interest for B_{ijk} is the main effect of factor B.