Name: __________ Practice Problems 2/13/2008

1. Let U be a binomial random variable with parameters $N = 3$ and p. Evaluate $\mathbb{E}\frac{1}{1+U}$.
Solution:

$$\begin{aligned}
\mathbb{E}\frac{1}{1+U} &= \sum_{k=0}^{N} \frac{1}{k+1}\binom{N}{k} p^k (1-p)^{N-k} \\
&= \sum_{k=0}^{N} \frac{N!}{(k+1)!(N-k)!} p^k (1-p)^{N-k} \\
&= \sum_{i=1}^{N+1} \frac{N!}{i!(N+1-i)!} p^{i-1} (1-p)^{N+1-i} \\
&= \frac{1}{p(N+1)} \sum_{i=1}^{N+1} \frac{(N+1)!}{i!(N+1-i)!} p^i (1-p)^{N+1-i} \\
&= \frac{1}{p(N+1)} \left(1 - (1-p)^{N+1}\right).
\end{aligned}$$

When $N = 3$, we have

$$\mathbb{E}\frac{1}{1+U} = \frac{1-(1-p)^4}{4p}.$$

2. Suppose X has binomial distribution with parameter N and $p = \frac{1}{2}$, and N and binomial distribution with parameters $M = 20$ and $q = \frac{1}{4}$. Evaluate $\mathbb{E}X$ and $\text{Var}(X)$.
Solution:

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|N)) = \mathbb{E}(Np) = Mpq = \frac{5}{2}.$$

$$\begin{aligned}
\text{Var}(X) &= \text{Var}(\mathbb{E}(X|N)) + \mathbb{E}(\text{Var}(X|N)) \\
&= \text{Var}(Np) + \mathbb{E}(Np(1-p) \\
&= Mq(1-q)p^2 + Mqp(1-p) \\
&= Mpq[p(1-q) + q(1-p)] \\
&= \frac{5}{4}.
\end{aligned}$$

3. A store that stocks a certain commodity uses the following ordering policy, if the supply at the the beginning of a time period is x, it orders $\begin{cases} 0 & x \geq 3 \\ 5-x & x < 3 \end{cases}$. The order is immediately filled. The daily demands are independent and equal j with probability $1/10$ for $j = 0, 1, ..., 9$. All demands that cannot be immediately met are lost. Let X_n denote the inventory level at the end of the n-th time period. Find the transition matrix of the Markov chain X_n.

Solution: The state space is $\{0, 1, 2, 3, 4, 5\}$. Given $X_n = i$. If $0 \leq i < 3$, then new order of $(5-i)$ is placed, and at the beginning of $(n+1)$th period, the inventory level will be 5. Thus, for $0 \leq j \leq 5$,

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(\text{ sell } 5-j) = \begin{cases} 1/10 & 0 \leq j \leq 4,\ 0 \leq i < 3 \\ 1/2 & j = 5, 0 \leq i < 3 \end{cases}.$$

If $3 \le i \le 5$, then at the beginning of $(n+1)$th period, the inventory level is i, thus,

$$\mathbb{P}(X_{n+1}=j|X_n=i)=\mathbb{P}(\text{ sell } i-j)=\begin{cases} 1/10 & 0\le j<i,\, 3\le i\le 5 \\ 1-i/10 & j=i,\, 3\le i\le 5 \\ 0 & j>i,\, 3\le i\le 5 \end{cases}.$$

Hence

$$P=\begin{pmatrix} 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.5 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.5 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.5 \\ 0.1 & 0.1 & 0.8 & 0 & 0 & 0 \\ 0.1 & 0.1 & 0.1 & 0.7 & 0 & 0 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.6 & 0 \end{pmatrix}.$$

4. Let X_n be a Markov chain with state space $\{0,1,2\}$ and the corresponding transition matrix

$$\begin{pmatrix} \alpha & 1-\alpha & 0 \\ \beta & 0 & 1-\beta \\ \alpha & \beta & 1-\alpha-\beta \end{pmatrix}.$$

If $X_0=0$, find $\mathbb{P}(X_2=1,X_0=0)$, and $\mathbb{P}(X_2>1,X_1=1,X_0=0)$.
Solution:

$$\begin{aligned} & \mathbb{P}(X_2=1,X_0=0) \\ = & \mathbb{P}(X_2=1,X_1=0,X_0=0)+\mathbb{P}(X_2=1,X_1=1,X_0=0)+\mathbb{P}(X_2=1,X_1=2,X_0=0) \\ = & P_{00}P_{01}+P_{01}P_{11}+P_{02}P_{21} \\ = & \alpha(1-\alpha). \end{aligned}$$

$$\mathbb{P}(X_2>1,X_1=1,X_0=0)=\mathbb{P}(X_2=1,X_1=1,X_0=0)=P_{01}P_{12}=(1-\alpha)(1-\beta).$$

5. Let ξ_n be i.i.d. random variables with $\mathbb{P}(\xi_1=0)=\alpha$ and $\mathbb{P}(\xi_1=1)=1-\alpha$. Let $X_0=0$ and $X_n=\xi_1+\xi_2+\cdots+\xi_n \pmod{5}$ for $n\ge 1$. Is X_n a Markov chain? If so, find its state space and the transition matrix.
Solution: The state space is $\{0,1,2,3,4\}$. Given $X_n=i$. If $i<4$, then

$$\mathbb{P}(X_{n+1}=i|X_n=i)=\mathbb{P}(\xi_{n+1}=0)=\alpha;\quad \mathbb{P}(X_{n+1}=i+1|X_n=i)=\mathbb{P}(\xi_{n+1}=1)=1-\alpha.$$

$X_n=i$. If $i=4$, then

$$\mathbb{P}(X_{n+1}=4|X_n=4)=\mathbb{P}(\xi_{n+1}=0)=\alpha;\quad \mathbb{P}(X_{n+1}=0|X_n=4)=\mathbb{P}(\xi_{n+1}=1)=1-\alpha.$$

Hence

$$P=\begin{pmatrix} \alpha & 1-\alpha & 0 & 0 & 0 \\ 0 & \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & \alpha & 1-\alpha & 0 \\ 0 & 0 & 0 & \alpha & 1-\alpha \\ 1-\alpha & 0 & 0 & 0 & \alpha \end{pmatrix}$$

6. Let X_n be a Markov chain with state space $\{0,1,2,3\}$ and the corresponding transition matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.3 & 0.3 & 0.4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(a) Starting state 1, find the mean time until absorption.
(b) If for each transient state it visits, it collects \$2 on an even-numbered state, or pays \$1 on an odd-numbered state. Find the mean total score.

Solution: (a) Let v_i be the mean duration if it starts at state i. Then $v_1 = 1 + 0.5v_2$, and $v_2 = 1 + 0.3v_1 + 0.3v_2$. Solving the system, we obtain $v_1 = 24/11$.

(b) Let s_i be the mean total score if it starts at state i. Then $s_1 = 0.5(2 + s_2)$, and $s_2 = 0.3(-1 + s_1) + 0.3(2 + s_2)$. Solving for s_1, we have $s_1 = 17/11$.
7-8: Page 175, Problems 7.2 and 7.4.
Solution: These are two homework problems.