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Exact value of the n-term approximation of a diagonal operator

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Abstract

This paper determines the exact value of the n-term approximation of a diagonal linear operator from l_p^M to l_q^M, $0 < p, q \le \infty$ using an elementary method.
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Keywords: n-term approximation; Diagonal operator

1. Introduction

Let $T: (x_1, x_2, \ldots, x_M) \mapsto (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_M x_M)$ be a diagonal linear operator from l_p^M to l_q^M, $0 < p, q \le \infty$. Following Stechkin [4], for $1 \le n \le M$, we define the n-term approximation of T as the quantity

$$\sigma_n(T) = \begin{cases} \sup\limits_{f \in B_p} \inf\limits_{\Gamma_n} \left(\sum\limits_{i \notin \Gamma_n} |\lambda_i f_i|^q \right)^{1/q}, & q < \infty \\ \sup\limits_{f \in B_p} \inf\limits_{\Gamma_n} \sup\limits_{i \notin \Gamma_n} |\lambda_i f_i|, & q = \infty \end{cases}$$

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where B_p is the unit ball of l_p^M, and Γ_n is an arbitrary subset of $\{1, 2, \ldots, M\}$ with n elements. Clearly, to determine the n-term approximation of a diagonal linear operator, we can assume that the λ_i are non-negative and non-increasing. Indeed, we do make this assumption for the rest of the paper.

We are interested in finding the exact value of $\sigma_n(T)$ for all $0 < p, q \le \infty$. When $0 < p \le q < \infty$, Stepanets [5] proved that

$$\sigma_n(T) = \max_{n<m\le M} \frac{(m-n)^{1/q}}{\left(\sum_{i=1}^{m} \lambda_i^{-p}\right)^{1/p}}.$$

Fang and Qian [1] gave a different proof for the case $p = q$ based on Ky Fan's minimax theorem [3]. It is therefore a natural question to ask what the exact value of $\sigma_n(T)$ is when $p > q$. Indeed, Fang and Qian [2] proved that when T is the identity operator from l_p^M to l_q^M, $\sigma_n(T) = (M-n)^{1/q} M^{-1/p}$ for $0 < q < p < \infty$, and made the following conjecture:

Conjecture 1 (*Fang and Qian [2]*). *For the diagonal operator $T: l_p^M \mapsto l_q^M$, $T(x_1, x_2, \ldots, x_M) = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_M x_M)$, $0 < q < p < \infty$,*

$$\sigma_n(T) = M^{1/q-1/p} \max_{n<m\le M} \left(\frac{m-n}{\sum_{i=1}^{m} \lambda_i^{-q}} \right)^{1/q}.$$

The goal of this paper is to answer this question by proving

Theorem 2. *Let $T: (x_1, x_2, \ldots, x_M) \mapsto (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_M x_M)$ be a diagonal linear operator from l_p^M to l_q^M, with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_M > 0$. For $1 \le n < M$,*

$$\sigma_n(T) = \begin{cases} \max_{n<m\le M} \dfrac{(m-n)^{1/q}}{\left(\sum_{i=1}^{m} \lambda_i^{-p}\right)^{1/p}}, & 0 < p \le q < \infty \\ \dfrac{1}{\left(\sum_{i=1}^{n+1} \lambda_i^{-p}\right)^{1/p}}, & 0 < p < q = \infty \\ \lambda_{n+1}, & p = q = \infty \\ \left(\dfrac{(m_0-n)^{p/(p-q)}}{\left(\sum_{i=1}^{m_0} \lambda_i^{-p}\right)^{q/(p-q)}} + \sum_{i=m_0+1}^{M} \lambda_i^{pq/(p-q)} \right)^{\frac{1}{q}-\frac{1}{p}}, & 0 < q < p < \infty \\ \left(\sum_{i=n+1}^{M} \lambda_i^{q} \right)^{1/q}, & 0 < q < p = \infty, \end{cases}$$

where m_0 is the largest integer m such that $n < m \le M$ and

$$(m-n)\lambda_m^{-p} \le \sum_{j=1}^{m} \lambda_j^{-p}.$$

From Theorem 2 it is clear that Conjecture 1 is not valid.

Remark 3. The result for the case $0 < p \le q \le \infty$ is due to Stepanets [5]. However, the proof given in this paper is much simpler. When $p = q < \infty$, we have

$$\sigma_n(T) = \max_{n<m\le M} \frac{(m-n)^{1/p}}{\left(\sum_{i=1}^{m} \lambda_i^{-p}\right)^{1/p}}.$$

On the other hand, by taking the limit $q \to p^-$ for the case $0 < q < p < \infty$, we can also obtain the exact value of $\sigma_n(T)$ for the case $p = q < \infty$, that is

$$\begin{aligned}
\sigma_n(T) &= \lim_{q\to p^-} \left(\frac{(m_0-n)^{p/(p-q)}}{\left(\sum_{i=1}^{m_0} \lambda_i^{-p}\right)^{q/(p-q)}} + \sum_{i=m_0+1}^{M} \lambda_i^{pq/(p-q)} \right)^{\frac{1}{q}-\frac{1}{p}} \\
&= \max \left\{ \frac{(m_0-n)^{1/p}}{\left(\sum_{i=1}^{m_0} \lambda_i^{-p}\right)^{1/p}}, \lambda_{m_0+1}, \lambda_{m_0+2}, \ldots, \lambda_M \right\} \\
&= \frac{(m_0-n)^{1/p}}{\left(\sum_{i=1}^{m_0} \lambda_i^{-p}\right)^{1/p}},
\end{aligned}$$

where the last equality follows from the definition of m_0. This last expression is more explicit.

2. Proof

1. Case $0 < p \le q < \infty$

We assume $p < q$ because the subcase $p = q$ can be handled by taking the limit $q \to p^+$. Because the supremum can be attained, there exists an f with $\sum_{i=1}^{M} |f_i|^p = 1$ such that

$$\sigma_n(T)^q = \sum_{i=1}^{M-n} |f_{\pi(i)}\lambda_{\pi(i)}|^q, \tag{1}$$

where $\{|f_{\pi(i)}\lambda_{\pi(i)}|\}$ is a non-decreasing arrangement of $\{|f_i\lambda_i|\}$.

First, we claim that for all $M-n < i \le M$,

$$|f_{\pi(i)}\lambda_{\pi(i)}| = |f_{\pi(M-n)}\lambda_{\pi(M-n)}|. \tag{2}$$

Indeed, because $\{|f_{\pi(i)}\lambda_{\pi(i)}|\}$ is a non-decreasing arrangement of $\{|f_i\lambda_i|\}$, we have

$$|f_{\pi(i)}\lambda_{\pi(i)}| \ge |f_{\pi(M-n)}\lambda_{\pi(M-n)}| \quad \text{for all } M-n < i \le M.$$

Suppose for some $M - n < i_0 \le M$

$$|f_{\pi(M-n)}\lambda_{\pi(M-n)}| = |f_{\pi(M-n+1)}\lambda_{\pi(M-n+1)}| = \cdots = |f_{\pi(i_0-1)}\lambda_{\pi(i_0-1)}| < |f_{\pi(i_0)}\lambda_{\pi(i_0)}|.$$

Denote $\alpha = \left(\sum_{i \ne i_0} |f_{\pi(i)}|^p + |f_{\pi(i_0-1)}|^p \lambda^p_{\pi(i_0-1)}\lambda^{-p}_{\pi(i_0)}\right)^{-1/p}$. Because

$$\sum_{i \ne i_0} |f_{\pi(i)}|^p + |f_{\pi(i_0-1)}|^p \lambda^p_{\pi(i_0-1)}\lambda^{-p}_{\pi(i_0)} < \sum_{i=1}^{M} |f_{\pi(i)}|^p = 1,$$

we have $\alpha > 1$. Define g such that

$$g_{\pi(i)} = \begin{cases} \alpha|f_{\pi(i)}|, & i \ne i_0 \\ \alpha|f_{\pi(i_0-1)}|\lambda_{\pi(i_0-1)}\lambda^{-1}_{\pi(i_0)}, & i = i_0. \end{cases}$$

Clearly, $\|g\|_p = \|f\|_p = 1$. Now, for this g, $\{g_{\pi(i)}\lambda_{\pi(i)}\}$ is a non-decreasing rearrangement of $\{g_i\lambda_i\}$. Thus,

$$\inf_{\Gamma_n} \sum_{i \notin \Gamma_n} |\lambda_i g_i|^q = \sum_{i=1}^{M-n} |g_{\pi(i)}\lambda_{\pi(i)}|^q.$$

Because $g_{\pi(i)} = \alpha|f_{\pi(i)}|$ for all $1 \le i \le M - n < i_0$, we have

$$\inf_{\Gamma_n} \sum_{i \notin \Gamma_n} |\lambda_i g_i|^q = \alpha^q \sum_{i=1}^{M-n} |f_{\pi(i)}\lambda_{\pi(i)}|^q = \alpha^q \sigma_n(T)^q.$$

This is impossible because $\alpha > 1$. Thus, for all $M - n < i \le M$, (2) holds.

Next, we claim that for all $1 \le i < M - n$, either $f_{\pi(i)} = 0$, or (2) holds. Suppose that for some $1 \le j_0 < M - n$, we have

$$0 < |f_{\pi(j_0)}\lambda_{\pi(j_0)}| < |f_{\pi(j_0+1)}\lambda_{\pi(j_0+1)}| = \cdots = |f_{\pi(M)}\lambda_{\pi(M)}|. \quad (3)$$

Consider the strictly convex function

$$F(x_1, x_2, \ldots x_{M-n}) := \sum_{i=1}^{M-n} x_i^{q/p} \lambda^q_{\pi(i)}.$$

On the convex domain

$$\left\{(x_1, x_2, \ldots, x_{M-n}) \in \mathbb{R}^{M-n} : \sum_{i=1}^{M-n} x_i \le 1 - \sum_{i=M-n+1}^{M} |f_{\pi(i)}|^p, \right.$$
$$\left. 0 \le x_i \le \left(|f_{\pi(M-n)}|\lambda_{\pi(M-n)}\lambda^{-1}_{\pi(i)}\right)^p \right\},$$

F attains its maximum only at an extreme point. Because by the assumption (3), the point

$$(|f_{\pi(1)}|^p, |f_{\pi(2)}|^p, \ldots, |f_{\pi(M-n)}|^p)$$

is inside the above convex set, but not an extreme point thereof, there exists an extreme point $(g^p_{\pi(1)}, g^p_{\pi(2)}, \ldots, g^p_{\pi(M-n)})$ with either $g_{\pi(i)} = 0$ or

$$g_{\pi(i)}\lambda_{\pi(i)} = |f_{\pi(M-n)}|\lambda_{\pi(M-n)},$$

such that

$$F(|f_{\pi(1)}|^p, |f_{\pi(2)}|^p, \ldots, |f_{\pi(M-n)}|^p) < F(g^p_{\pi(1)}, g^p_{\pi(2)}, \ldots, g^p_{\pi(M-n)}).$$

By defining $g_{\pi(i)} = |f_{\pi(i)}|$ for $M - n < i \le M$, we have $\|g\|_p \le 1$ and

$$\begin{aligned}\inf_{\Gamma_n} \sum_{i \notin \Gamma_n} |\lambda_i g_i|^q &= \sum_{i=1}^{M-n} |g_{\pi(i)}\lambda_{\pi(i)}|^q = F(g^p_{\pi(1)}, g^p_{\pi(2)}, \ldots, g^p_{\pi(M-n)}) \\ &> F(|f_{\pi(1)}|^p, |f_{\pi(2)}|^p, \ldots, |f_{\pi(M-n)}|^p) = \sigma_n(T)^q,\end{aligned}$$

which is impossible. Hence, either $f_{\pi(i)} = 0$ or $|f_{\pi(i)}\lambda_{\pi(i)}| = |f_{\pi(M-n)}\lambda_{\pi(M-n)}|$.

Therefore, we have a constant $c > 0$ and an index set $I \subset \{1, 2, \ldots, M\}$ such that $|f_i| = c\lambda_i^{-1}$ for $i \in I$ and $f_i = 0$ for $i \notin I$. Together with $\sum_{i=1}^M |f_i|^p = 1$, we have

$$|f_i| = \begin{cases} \lambda_i^{-1} \left(\sum_{i \in I} \lambda_i^{-p} \right)^{-1/p}, & i \in I \\ 0, & i \notin I. \end{cases}$$

For this f we have

$$\sum_{i=1}^{M-n} |f_{\pi(i)}\lambda_{\pi(i)}|^q = \frac{|I| - n}{\left(\sum_{i \in I} \lambda_i^{-p} \right)^{q/p}}.$$

Because the λ_i are positive and non-increasing, the optimal value is attained if I is of the form $I = \{1, 2, \ldots, m\}$ for some $m > n$. Hence

$$\sigma_n(T)^q = \sup_{m>n} \frac{m - n}{\left(\sum_{i=1}^{m} \lambda_i^{-p} \right)^{q/p}},$$

as desired.

2. Case $q < p < \infty$

It is possible to use the same approach as in the previous case. However, the following proof is even simpler. In fact, the only trick is the simple fact that

$$\begin{aligned}&\inf \left\{ \sum_{i=1}^M a_i \delta_i : \delta_i \in \{0, 1\} : \sum_{i=1}^M \delta_i = M - n \right\} \\ &\quad = \inf \left\{ \sum_{i=1}^M a_i \eta_i : \eta_i \in [0, 1], \sum_{i=1}^M \eta_i = M - n \right\}\end{aligned}$$

which is true because a linear function on a convex domain attains its extreme value at an extreme point. Using this simple fact, we have

$$\begin{aligned}\sigma_n(T)^q &= \sup_{f \in B_p} \inf \left\{ \sum_{i=1}^M |f_i|^q \lambda_i^q \delta_i : \delta_i \in \{0, 1\}, \sum_{i=1}^M \delta_i = M - n \right\} \\ &= \sup_{f \in B_p} \inf \left\{ \sum_{i=1}^M |f_i|^q \lambda_i^q \eta_i : \eta_i \in [0, 1], \sum_{i=1}^M \eta_i = M - n \right\}.\end{aligned}$$

Applying Hölder's inequality gives

$$\sigma(T)^q = \sup_{f\in B_p} \inf\left\{\sum_{i=1}^M |f_i|^q \lambda_i^q \eta_i : \eta_i \in [0,1], \sum_{i=1}^M \eta_i = M-n\right\}$$
$$\le \inf\left\{\left(\sum_{i=1}^M \lambda_i^{pq/(p-q)} \eta_i^{p/(p-q)}\right)^{(p-q)/p} : \sum_{i=1}^M \eta_i = M-n; 0 \le \eta_i \le 1\right\}. \quad (4)$$

We define η_i such that

$$\eta_i = \begin{cases} \dfrac{(m_0-n)\lambda_i^{-p}}{\sum_{j=1}^{m_0} \lambda_j^{-p}}, & 1 \le i \le m_0 \\ 1, & m_0 < i \le M, \end{cases} \quad (5)$$

where m_0 is defined in the statement of the theorem. Clearly, $\eta_i \in [0,1]$ for all $1 \le i \le M$. By choosing η_i as defined in (5), we have from (4) that

$$\sigma_n(T)^q \le \left[\frac{(m_0-n)^{p/(p-q)}}{\left(\sum_{i=1}^{m_0} \lambda_i^{-p}\right)^{q/(p-q)}} + \sum_{i=m_0+1}^M \lambda_i^{pq/(p-q)}\right]^{(p-q)/p}.$$

To prove the other direction, we choose

$$f_i = \begin{cases} L^{-1/p} K^{1/(p-q)} \lambda_i^{-1}, & 1 \le i \le m_0 \\ L^{-1/p} \lambda_i^{q/(p-q)}, & m_0+1 \le i \le M, \end{cases} \quad (6)$$

where

$$K = (m_0-n)\left(\sum_{i=1}^{m_0} \lambda_i^{-p}\right)^{-1} \quad \text{and} \quad L = \frac{(m_0-n)^{p/(p-q)}}{\left(\sum_{i=1}^{m_0} \lambda_i^{-p}\right)^{q/(p-q)}} + \sum_{i=m_0+1}^M \lambda_i^{pq/(p-q)}.$$

It is easy to check that

$$\sum_{i=1}^M |f_i|^p = 1.$$

Thus,

$$\sigma_n(T)^q \ge \inf\left\{\sum_{i=1}^M |f_i|^q \lambda_i^q \eta_i : \eta_i \in [0,1], \sum_{i=1}^M \eta_i = M-n\right\}$$
$$= L^{-q/p} \inf\left\{K^{q/(p-q)} \sum_{i=1}^{m_0} \eta_i + \sum_{m_0+1}^M \lambda_i^{\frac{pq}{p-q}} \eta_i : \eta_i \in [0,1], \sum_{i=1}^M \eta_i = M-n\right\}.$$

Because the definition of m_0 implies

$$\lambda_{m_0+1}^{-p} > \frac{\sum_{j=1}^{m_0} \lambda_j^{-p}}{m_0 - n} = K^{-1},$$

we have

$$\lambda_i^{pq/(p-q)} \leq \lambda_{m_0+1}^{pq/(p-q)} < K^{q/(p-q)}$$

for all $m_0 + 1 \leq i \leq M$. Clearly, the infimum above is attained when $\eta_{m_0+1} = \cdots = \eta_M = 1$. Consequently, we have

$$\begin{aligned} \sigma_n(T)^q &\geq L^{-q/p} \left(K^{q/(p-q)}(m_0 - n) + \sum_{i=m_0+1}^{M} \lambda_i^{\frac{pq}{p-q}} \right) \\ &= L^{(p-q)/p}. \end{aligned}$$

Therefore

$$\sigma_n(T) = L^{1/q-1/p} = \left(\frac{(m_0 - n)^{p/(p-q)}}{\left(\sum_{i=1}^{m_0} \lambda_i^{-p} \right)^{q/(p-q)}} + \sum_{i=m_0+1}^{M} \lambda_i^{pq/(p-q)} \right)^{\frac{1}{q} - \frac{1}{p}},$$

as desired.

3. Remaining cases

The proofs for the remaining cases are straightforward.

Remark 4. The above proof of the theorem also shows that for $q \leq p$

$$\begin{aligned} &\sup_{f \in B_p} \inf \left\{ \sum_{i=1}^{M} |f_i|^q \lambda_i^q \eta_i : \sum_{i=1}^{M} \eta_i = M - n; 0 \leq \eta_1, \eta_2, \ldots, \eta_M \leq 1 \right\} \\ &\quad = \inf \left\{ \sup_{f \in B_p} \sum_{i=1}^{M} |f_i|^q \lambda_i^q \eta_i : \sum_{i=1}^{M} \eta_i = M - n; 0 \leq \eta_1, \eta_2, \ldots, \eta_M \leq 1 \right\}, \end{aligned}$$

which can be compared with the minimax theorem of [1] (Theorem 1). This may be of independent interest.

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