Exact L^2 small balls of Gaussian processes

F. Gao[*], J. Hannig[†] T.-Y. Lee[‡] and F. Torcaso[§]

August 6, 2011

Abstract

We prove a comparison theorem, extending Li [6], and develop a complex-analytic approach to treat L^2 small ball probabilities of Gaussian processes. We demonstrate the techniques for the m-times integrated Brownian motions, and in examples where one can not apply Li's comparison theorem.

[*]Department of Mathematics, University of Idaho, Moscow, ID 83844-1103, fuchang@uidaho.edu

[†]Department of Statistics, Colorado State University, Ft. Collins, CO 80523-1877, hannig@stat.colostate.edu

[‡]Mathematics Department, University of Maryland, College Park, MD 20742-4015, (301) 405-5086, tyl@math.umd.edu

[§]Department of Mathematical Sciences, Johns Hopkins University, Baltimore, MD 21218-2682, torcaso@mts.jhu.edu

1 Introduction

Let $X(t)$ for $0 \le t \le 1$ be a centered Gaussian process with covariance kernel $K(s,t)$. We are interested in the small ball probability of $\|X\|$, i.e., $P(\|X\| \le \varepsilon)$ as ε tends to 0, where $\|\cdot\|$ denotes the norm in $L^2[0,1]$. A Karhunen-Loève expansion of $X(t)$ yields $\|X\|^2 = \sum_{n=1}^{\infty} a_n \xi_n^2$ where $\{\xi_n\}_{n=1}^{\infty}$ is a sequence of independent standard Gaussian random variables and $\{a_n\}_{n=1}^{\infty}$ is the sequence of eigenvalues of the corresponding covariance operator $\mathcal{A}$:

$$\mathcal{A}\zeta(t) \equiv \int_0^1 K(s,t)\zeta(s)\,ds = a\zeta(t) \tag{1}$$

where $\zeta \in L^2[0,1]$.

The small ball probability for Gaussian processes in L^2 norm, equivalently, for r.v. $A = \sum_{n=1}^{\infty} a_n \xi_n^2$ can be expressed as:

$$P(\|X\| \le \varepsilon) = P\left(A \le \varepsilon^2\right) \sim \left(-2\pi\gamma^2 h''(\gamma)\right)^{-1/2} \exp\left\{\gamma h'(\gamma) - h(\gamma)\right\} \quad \text{as} \quad \varepsilon \to 0 \tag{2}$$

where $h(\gamma) = \frac{1}{2}\sum_{n=1}^{\infty} \log(1 + 2a_n\gamma)$ and $\varepsilon^2 = h'(\gamma)$. Here and in what follows $x(\varepsilon) \sim y(\varepsilon)$ as $\varepsilon \to 0$ means $\lim_{\varepsilon\to 0} x(\varepsilon)/y(\varepsilon) = 1$. This result was first proved by Sytaya [8] using saddle point approximations to the Laplace transform. See [1] for a nice probabilistic proof. Notice that (2) only depends on X through the eigenvalues a_n. However, even in the event an explicit expression is known for the a_n, one will typically encounter difficulties applying (2). For instance, finding expressions for the function $h(\gamma)$ and determining the implicit relation $\gamma = \gamma(\varepsilon)$ are two essential difficulties that arise. In fact, except in some very special cases, the eigenvalues a_n cannot be explicitly computed. Nevertheless, in instances where we have suitable approximations b_n to the eigenvalues a_n, we have the following result due to Li [6].

Throughout this article we assume, without loss of generality, that sequences $a_n, b_n, \ldots,$ are positive, non-increasing, and summable; otherwise, r.v. $\sum_{n=1}^{\infty} a_n \xi_n^2$ is ∞ with probability 1.

Theorem 1.

$$P\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \le \varepsilon^2\right) \sim \left(\prod_{n=1}^{\infty} b_n/a_n\right)^{1/2} P\left(\sum_{n=1}^{\infty} b_n \xi_n^2 \le \varepsilon^2\right) \tag{3}$$

as $\varepsilon \to 0$, *provided*

$$\sum_{n=1}^{\infty} \left|1 - \frac{a_n}{b_n}\right| < \infty.$$

The above theorem of Li's [6] has been a useful tool to study L^2-small ball probabilities. The difficulty in using the theorem is in checking the absolute summability condition since one needs to know the asymptotic behavior of a_n well.

The goal of this paper is to first remove this condition from Theorem 1. Then we use method from complex analysis to show how one can evaluate the infinite product $\prod_{n=1}^{\infty} b_n/a_n$ directly without necessarily knowing each individual a_n and b_n.

The first result is

Theorem 2.

$$P\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \le \varepsilon^2\right) \sim \left(\prod_{n=1}^{\infty} b_n/a_n\right)^{1/2} P\left(\sum_{n=1}^{\infty} b_n \xi_n^2 \le \varepsilon^2\right)$$

as $\varepsilon \to 0$, *provided the infinite product* $\prod_{n=1}^{\infty} b_n/a_n$ *converges.*

Remark 1. In Section 3, we will see examples where Theorem 2 can be applied while Theorem 1 cannot. However, the main point of Theorem 2 is not only to weaken the assumption on the sequence $\{a_n\}$ and $\{b_n\}$, but also to make the direct computation in the following theorem possible.

The second result is

Theorem 3. *Set $\rho_n = 1/a_n$ and $\nu_n = 1/b_n$. Suppose f and g are entire functions such that the ρ_n are the only zeros of f, and the ν_n are the only zeros of g. Assume for simplicity that ρ_n, ν_n are simple zeros for all sufficiently large n. If*

$$\max_{-\pi\le\theta\le\pi}\left|\frac{f(r_n e^{i\theta})}{g(r_n e^{i\theta})}-1\right| \to 0 \quad \text{as} \quad n\to\infty \tag{4}$$

for some sequence $\{r_n\}$ with $\nu_n < r_n < \nu_{n+1}$ for all sufficiently large n, then

$$P\left(\sum_{n=1}^{\infty} a_n\xi_n^2 \le \varepsilon^2\right) \sim \left(\frac{|f(0)|}{|g(0)|}\right)^{1/2} P\left(\sum_{n=1}^{\infty} b_n\xi_n^2 \le \varepsilon^2\right) \quad \text{as} \quad \varepsilon\to 0. \tag{5}$$

Let us remark that to require f and g to be entire functions is not a big restriction. While f is usually derived naturally from (1), the entire function g is typically chosen to be simple, for example, the leading order term of f as $|z|\to\infty$ along some sequence.

Remarkably, Theorem 3 allows us to evaluate the infinite product in (3) exactly even if one does not know the a_n. The novelty in the proof of Theorem 3 is the use of certain complex analytic methods. The proof uses Rouché's theorem [7] (page 225): if f and g are entire and $|f(z)| \ge |f(z)-g(z)|$ on some simple closed contour, then f and g have the same number of zeros counting multiplicity inside this contour. Another theorem we will need is Jensen's formula [7] (page 307): if g is an entire function such that $g(0) \neq 0$, then, for any $r > 0$,

$$|g(0)| \prod_{n=1}^{L} \frac{r}{|z_n|} = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left|g\left(re^{i\theta}\right)\right| d\theta \right\}.$$

where $z_1, z_2, \ldots, z_L$ are the zeros of g counting multiplicity inside $\Gamma = \{z : |z| = r\}$.

The rest of the paper is organized as follows: in Section 2, we prove Theorem 2; in Section 3, we present some examples using Theorem 2; in Section 4, we prove Theorem 3 using Theorem 2; in Section 5, we apply Theorem 3 to obtain the exact small ball probability for all generalized m-times integrated Brownian motions (see [2]). With Theorem 3 we are able to show that the exact small ball probability for the usual m-times integrated Brownian motion is larger than that of the m-times Euler integrated Brownian motion. This suggests a possible stochastic domination between the L^2-norm of these processes (see the remarks at the end of Section 5).

2 Proof of Theorem 2

In this section, we prove Theorem 2. We begin with a series of lemmas that will be useful for the proof.

Lemma 1. *If $a_n > 0$ and $\sum a_n < \infty$. Then $\prod_{n=1}^{\infty}(1 + 2a_n x) \in C^{\infty}(0, \infty)$.*

Proof. Let $A(x) = \log \prod_{n=1}^{\infty}(1 + 2a_n x)$. We prove that $A \in C^{\infty}(0, \infty)$. Formally, we can write

$$A^{(k)}(x) = \sum_{n=1}^{\infty} \frac{(-1)^{k-1}(k-1)!(2a_n)^k}{(1 + 2a_n x)^k} \quad \text{for} \quad k = 1, 2, \ldots$$

The equality becomes true if the series converges uniformly on $(0, \infty)$. However, this is easily verified. □

Lemma 2. *Let $\{c_n\}$ be a monotone sequence with $0 < c_n \leq 1$ for all $n \geq 1$. Suppose $\sum_{n=1}^{\infty} r_n$ converges. Then for any $N < M \leq \infty$,*

$$\left| \sum_{n=N}^{M} c_n r_n \right| \leq 2 \sup_{n \geq N} \left| \sum_{k=n}^{\infty} r_k \right|.$$

This is just a consequence of Abel's summation formula, and we omit the proof.

Lemma 3. *Let $A(x) = \log \prod_{n=1}^{\infty}(1 + 2a_n x)$ and $B(x) = \log \prod_{n=1}^{\infty}(1 + 2b_n x)$, where $a_n, b_n > 0$, $\sum a_n < \infty$ and $\sum b_n < \infty$, $a_n \geq a_{n+1}$ and $b_n \geq b_{n+1}$ for all n. Suppose further that $\prod_{n=1}^{\infty} a_n/b_n$ converges, then*

$$\lim_{x\to\infty} [A(x) - B(x)] = \log \prod_{n=1}^{\infty} \frac{a_n}{b_n}.$$

Proof. By assumption, we can write $a_n = b_n(1+r_n)$, where $\sum_{n=1}^{\infty} \log(1+r_n)$ converges. Then

$$\begin{aligned} A(x) - B(x) &= \lim_{k\to\infty} \log \prod_{n=1}^{k} \frac{1+2a_n x}{1+2b_n x} \\ &= \lim_{k\to\infty} \log \prod_{n=1}^{k} \left(1 + \frac{2b_n x}{1+2b_n x} r_n\right) \\ &= \log \prod_{n=1}^{N} \left(1 + \frac{2b_n x}{1+2b_n x} r_n\right) + \lim_{k\to\infty} \log \prod_{n=N+1}^{k} \left(1 + \frac{2b_n x}{1+2b_n x} r_n\right). \end{aligned} \tag{6}$$

Note that for $0 < c < 1$, and $|x| < 1$, $\log(1 + cx) \geq c \log(1 + x)$. Choose N large enough so that $|r_n| < 1$ for all $n > N$. Then the second term on the right is bounded from below by

$$\liminf_{k\to\infty} \sum_{n=N+1}^{k} \frac{2b_n x}{1+2b_n x} \log(1 + r_n).$$

Applying Lemma 2 for the non-increasing sequence $c_n = 2b_nx/(1+2b_nx)$, we have

$$\sum_{n=N+1}^{k} \frac{2b_nx}{1+2b_nx}\log(1+r_n) \geq -2\sup_{n\geq N+1}\left|\sum_{k=n}^{\infty}\log(1+r_k)\right|.$$

Letting $x \to \infty$ and then $N \to \infty$ on (6), we obtain

$$\liminf_{x\to\infty}[A(x)-B(x)] \geq \log\prod_{n=1}^{\infty}(1+r_n) = \log\prod_{n=1}^{\infty}\frac{a_n}{b_n}.$$

By considering $B(x)-A(x)$ in the same way, we complete the proof. □

Lemma 4. *Let $A(x)$, $B(x)$, a_n and b_n be as in Lemma 3. Then*

$$xA'(x) - xB'(x) \to 0 \quad \textit{and} \quad x^2A''(x) - x^2B''(x) \to 0 \quad \textit{as} \quad x \to \infty.$$

Proof. We use the same notation r_n as in the proof of Lemma 3. By Lemma 1, $A(x)$ and $B(x)$ are differentiable, and

$$\begin{aligned} xA'(x) - xB'(x) &= \sum_{n=1}^{\infty}\left(\frac{2a_nx}{1+2a_nx} - \frac{2b_nx}{1+2b_nx}\right) \\ &\geq \sum_{n=1}^{N}\frac{1}{1+2a_nx}\cdot\frac{2b_nx}{1+2b_nx}\cdot r_n + \sum_{n=N+1}^{\infty}\frac{1}{1+2a_nx}\cdot\frac{2b_nx}{1+2b_nx}\cdot\log(1+r_n). \end{aligned} \tag{7}$$

Applying lemma 2 twice: first for the non-decreasing sequence $c_n = 1/(1+2a_nx)$, and then for the non-increasing sequence $\tilde{c}_n = 2b_nx/(1+2b_nx)$, we have

$$\sum_{n=N+1}^{\infty}\frac{1}{1+2a_nx}\cdot\frac{2b_nx}{1+2b_nx}\cdot\log(1+r_n) \geq -4\sup_{n\geq N+1}\left|\sum_{k=n}^{\infty}\log(1+r_k)\right|$$

Letting $x \to \infty$ and then $N \to \infty$ on (7), we obtain

$$\liminf_{x\to\infty}[xA'(x) - xB'(x)] \geq 0.$$

By considering $xB'(x) - xA'(x)$ in the same way, we have

$$\lim_{x\to\infty} [xA'(x) - xB'(x)] = 0. \tag{8}$$

Similarly, we write,

$$\begin{aligned} x^2B''(x) - x^2A''(x) \ \geq \ & \sum_{n=1}^{\infty} \frac{1}{1+2a_nx} \cdot \frac{2a_nx}{1+2a_nx} \cdot \frac{2b_nx}{1+2b_nx} \cdot \log(1+r_n) \\ & + \sum_{n=1}^{\infty} \frac{1}{1+2a_nx} \cdot \left(\frac{2b_nx}{1+2b_nx}\right)^2 \cdot \log(1+r_n). \end{aligned}$$

Applying the same argument as in the proof of (8) to each of the two series gives $x^2A''(x) - x^2B''(x) \to 0$. □

Lemma 5. *Let $A(x)$, $B(x)$, a_n and b_n be as in Lemma 3. Moreover, if $y = y(x)$ is chosen to satisfy $A'(x) = B'(y)$, then*

$$x \sim y \quad \textit{and} \quad [xA'(x) - A(x)] - [yB'(y) - B(y)] \to \log\left(\prod_{n=1}^{\infty} \frac{a_n}{b_n}\right) \qquad \textit{as } x \to \infty.$$

Proof. By Lemma 3 and Lemma 4, We have

$$[xA'(x) - A(x)] - [xB'(x) - B(x)] \to \log\left(\prod_{n=1}^{\infty} \frac{a_n}{b_n}\right) \qquad \text{as } x \to \infty.$$

Thus, it is enough to show that

$$[xB'(x) - B(x)] - [yB'(y) - B(y)] \to 0 \qquad \text{as } x \to \infty.$$

To this end, we first observe that $A'(x) = B'(y)$ implies $x \sim y$. In fact, by Lemma 4, we have

$$xB'(y) - xB'(x) = xA'(x) - xB'(x) \to 0 \tag{9}$$

as $x \to \infty$ (and consequently $y \to \infty$). On the other hand,

$$|xB'(y) - xB'(x)| = \left|1 - \frac{x}{y}\right| \sum_{n=1}^{\infty} \frac{4b_n^2 xy}{(1+2b_n x)(1+2b_n y)} \geq \frac{1}{4}\left|1 - \frac{x}{y}\right| \tag{10}$$

for $x, y > 1/2b_1$. Thus, $1 - x/y \to 0$ as $x, y \to \infty$, that is $x \sim y$.

To prove that

$$[xB'(x) - B(x)] - [yB'(y) - B(y)] \to 0 \quad \text{as } x \to \infty,$$

we use the Mean Value Theorem,

$$[xB'(x) - B(x)] - [yB'(y) - B(y)] = (x - y)\theta B''(\theta)$$

for some θ between x and y.

Note that for $x, y > 1/2b_1$,

$$\begin{aligned}
|(y-x)\theta B''(\theta)| &= \left|\sum_{n=1}^{\infty} \frac{4b_n^2 (y-x)\theta}{(1+2b_n\theta)^2}\right| \\
&= \left|\sum_{n=1}^{\infty} \frac{4b_n^2 xy}{(1+2b_n x)(1+2b_n y)} \left(1 - \frac{x}{y}\right) \frac{\theta(1+2b_n x)(1+2b_n y)}{x(1+2b_n\theta)^2}\right| \\
&\leq \left(1 + \frac{\theta}{x}\right)\left(1 + \frac{y}{\theta}\right)\left|1 - \frac{x}{y}\right| \sum_{n=1}^{\infty} \frac{4b_n^2 xy}{(1+2b_n x)(1+2b_n y)} \\
&= \left(1 + \frac{\theta}{x}\right)\left(1 + \frac{y}{\theta}\right)|xB'(y) - xB'(x)| \to 0
\end{aligned}$$

as $x \to \infty$, where in the last step, we used (9), (10) and the following simple observation: If $0 < c < d < e$ then $(1 + ce)/(1 + cd) \leq (1 + e/d)$. The lemma follows. □

Lemma 6. *Let $A(x)$, $B(x)$, a_n and b_n be as in Lemma 3. If $x \sim y$ as $x, y \to \infty$, then $x^2 A''(x)/y^2 B''(y) \to 1$ as $x \to \infty$.*

Proof. Note that

$$\frac{A''(x)}{B''(x)} - 1 = \frac{x^2 A''(x) - x^2 B''(x)}{x^2 B''(x)}$$

The numerator goes to 0 as x goes to ∞ by Lemma 4, while the denominator

$$x^2 B''(x) = -\sum_{n=1}^{\infty} \frac{4b_n^2 x^2}{(1+2b_n x)^2}$$

goes to $-\infty$ as $x \to \infty$. Thus,

$$A''(x)/B''(x) \to 1 \quad \text{as} \quad x \to \infty. \tag{11}$$

To conclude the proof calculate

$$\begin{aligned}
&\left|x^2 B''(x) - y^2 B''(y)\right| \\
&\quad = \left| \sum_{n=1}^{\infty} \frac{4b_n^2 x^2}{(1+2b_n x)^2} \left(1 - \frac{y}{x}\right) \left[\left(1 + \frac{y}{x}\right) \frac{1}{1+2b_n y} + \left(1 - \frac{y}{x}\right) \frac{2b_n y}{(1+2b_n y)^2} \right] \right| \\
&\quad \le \left|1 - \frac{y}{x}\right| 2 \left(1 + \frac{y}{x}\right) \sum_{n=1}^{\infty} \frac{4b_n^2 x^2}{(1+2b_n x)^2} = 2 \left|1 - \frac{y}{x}\right| \left(1 + \frac{y}{x}\right) \left|x^2 B''(x)\right|
\end{aligned}$$

and therefore

$$\left|1 - \frac{y^2 B''(y)}{x^2 B''(x)}\right| \le 2 \left|1 - \frac{y}{x}\right| \left(1 + \frac{y}{x}\right) \to 0 \quad \text{as} \quad x \to \infty,$$

which together with (11) implies the statement of the lemma. □

Proof of Theorem 2. By the Sytaya's result (2), we have

$$\frac{P\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \le \varepsilon^2\right)}{P\left(\sum_{n=1}^{\infty} b_n \xi_n^2 \le \varepsilon^2\right)} = \left(\frac{x^2 A''(x)}{y^2 B''(y)}\right)^{-1/2} \exp\left\{[xA'(x) - A(x)] - [yB'(y) - B(y)]\right\},$$

where A and B as in Lemma 1; x and y are defined by $\varepsilon^2 = A'(x) = B'(y)$. Theorem 2 follows from Lemma 5 and Lemma 6. □

3 Examples

In this section, we present some examples of situation where Theorem 2 can be applied but Theorem 1 cannot.

First, let us look at the following simple example. Set $a_n = 1/(n+1)^2$, and for $m \geq 1$, $b_{2m-1} = b_{2m} = [(2m-1)(2m)]^{-1}$. Then clearly,

$$\prod_{n=1}^{\infty} \frac{a_n}{b_n} = 1.$$

Therefore, by Theorem 2 we have

$$P\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \leq \varepsilon^2\right) \sim P\left(\sum_{n=1}^{\infty} b_n \xi_n^2 \leq \varepsilon^2\right) \quad \text{as } \varepsilon \to 0^+.$$

However,

$$\sum_{n=1}^{\infty} \left|1 - \frac{a_n}{b_n}\right| = \sum_{m=1}^{\infty} \frac{1}{2m} = \infty.$$

Thus, Theorem 1 is not applicable.

Our second example is the following small ball probability studied by Hoffmann-Jørgensen, Shepp & Dudley [4].

$$P\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \leq \varepsilon^2\right)$$

where ξ_n are i.i.d. standard normal random variables, and

$$a_n = \left(2\left\lfloor\frac{n+1}{2}\right\rfloor - 1\right)^{-2},$$

$\lfloor x \rfloor$ is the greatest integer function (also called the floor function).

We will obtain this small ball probability by comparing it with the L^2 small ball probability of Brownian motion on $[0,1]$ and by using Theorem 2. It is easy to see that

Theorem 1 does not work for this problem since the absolute summability condition is not satisfied.

Let X be the Brownian motion on $[0,1]$. It is known that

$$P(\|X\|_2^2 \leq \varepsilon^2) \sim \frac{4}{\sqrt{\pi}}\varepsilon \exp\left(-\frac{1}{8\varepsilon^2}\right).$$

(Also, see Theorem 4 with $m = 0$.)

On the other hand, it is known that the eigenvalues are $[(n-1/2)\pi]^{-2}$. Thus, if we let $b_n = (n-1/2)^{-2}$, we have

$$P\left(\sum_{n=1}^{\infty} b_n \xi_n^2 \leq \varepsilon^2\right) = P(\|X\|_2^2 \leq \varepsilon^2/\pi^2) \sim \frac{4}{\pi\sqrt{\pi}}\varepsilon \exp\left(-\frac{\pi^2}{8\varepsilon^2}\right),$$

It is easy to see that

$$\left(\prod_{n=1}^{\infty} \frac{a_n}{b_n}\right)^{1/2} = \prod_{k=1}^{\infty}\left(1 - \frac{1}{4(2k-1)^2}\right) = \frac{1}{\sqrt{2}}.$$

Thus, by Theorem 2 we have

$$P\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \leq \varepsilon^2\right) \sim 4\sqrt{2}\pi^{-3/2}\varepsilon \exp\left(-\frac{\pi^2}{8\varepsilon^2}\right)$$

and this is what was obtained in [4].

Certainly the added benefits of Theorem 2 are not restricted to the above two examples. For example, by using Theorem 2, Li and Torcaso [5] are able to obtain the exact L^2 small ball probability for the integrated Brownian sheet. However, our interest is in its application to Theorem 3 in the next section. Theorem 3 enables us compute the exact small ball probability, without knowing the individual eigenvalues.

4 Proof of Theorem 3

Now we turn our attention to the proof of Theorem 3. As mentioned in the introduction we will use Rouché's theorem and Jensen's formula. The statement of these theorems can be found in any complex analysis textbook e.g., [7].

Proof of Theorem 3. By Theorem 2 we have

$$P\left(\|X\| \le \epsilon\right) \sim \left(\prod_{n=1}^{\infty} b_n/a_n\right)^{1/2} P\left(\|Y\| \le \epsilon\right) \quad \text{as} \quad \varepsilon \to 0,$$

provided that the infinite product converges. Thus, all we need to show is that the infinite product converges to $|f(0)/g(0)|$.

It follows from (4) that there exists n_0 such that for $n > n_0$, r_n is not equal to any of the ρ_n and ν_n. Furthermore, since the limit in (4) is uniform, there exists $n_1 > n_0$ such that for all $n > n_1$ and all $\theta \in [-\pi, \pi)$

$$|g(r_n e^{i\theta})| > |g(r_n e^{i\theta}) - f(r_n e^{i\theta})|.$$

By Rouché's theorem, $f(z)$ and $g(z)$ have the same number of zeros inside the circle $|z| = r_n$ for all $n > n_1$. Moreover the assumption $\nu_n < r_n < \nu_{n+1}$ implies that the number of zeros is n. Without loss of generality we assume $n_1 = 0$.

Noting that $\rho_j, \nu_j > 0$ we have by Jensen's formula

$$|f(0)| \prod_{j=1}^{n} \frac{r_n}{\rho_j} = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(r_n e^{i\theta})| d\theta\right) \tag{12}$$

and

$$|g(0)| \prod_{j=1}^{n} \frac{r_n}{\nu_j} = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|g(r_n e^{i\theta})| d\theta\right). \tag{13}$$

Upon dividing (13) by (12) we obtain

$$\frac{|g(0)|}{|f(0)|}\prod_{j=1}^{n}\frac{\rho_j}{\nu_j} = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\frac{|g(r_ne^{i\theta})|}{|f(r_ne^{i\theta})|}\,d\theta\right). \tag{14}$$

Letting $n \to \infty$, the righthand side of (14) converges to 1 and

$$\prod_{n=1}^{\infty}\frac{b_n}{a_n} = \prod_{n=1}^{\infty}\frac{\rho_n}{\nu_n} = \frac{|f(0)|}{|g(0)|}.$$

□

5 An Application: m-times Integrated Brownian motions

For integer $m \geq 0$, let $X(t)$ for $0 \leq t \leq 1$ be the usual m-times Integrated Brownian motion:

$$X(t) = \int_0^t\int_0^{s_m}\int_0^{s_{m-1}}\cdots\int_0^{s_2} B(s_1)\,ds_1\,ds_2\,\cdots\,ds_m \tag{15}$$

where $B(t)$ is a standard Brownian motion. Clearly, $E\left(X(t)\right) = 0$ for all t. It is not difficult to show that this process has the covariance kernel

$$K(s,t) = E\left(X(s)X(t)\right) = \frac{1}{(m!)^2}\int_0^{s\wedge t}(s-u)^m(t-u)^m\,du.$$

If we denote by $\mathcal{A}$ the associated covariance operator, then $\mathcal{A}$ enjoys the properties of being positive, compact and self-adjoint. Thus, the spectrum for $\mathcal{A}$ is well known to be positive, discrete, real and tends to 0, and the corresponding eigenfunctions form a complete orthonormal basis for $L^2[0,1]$. By successively differentiating (1) $2m+2$ times we see the problem is equivalent to the following higher order Sturm-Liouville problem:

$$a\zeta^{(2m+2)}(t) = (-1)^{m+1}\zeta(t), \quad 0 < t < 1 \tag{16}$$

$$\zeta(0) = \zeta'(0) = \cdots = \zeta^{(m)}(0) = \zeta^{(m+1)}(1) = \cdots = \zeta^{(2m)}(1) = \zeta^{(2m+1)}(1) = 0. \tag{17}$$

What is interesting from a probabilistic point of view is that a large class of m-times integrated Brownian motions can be developed from the Sturm-Liouville problem (16) by permuting the evaluation points in the boundary conditions (17) in the following manner:

$$\begin{aligned} \zeta(t_0) = \zeta'(t_1) = \cdots = \zeta^{(m)}(t_m) = \zeta^{(m+1)}(t_{m+1}) = \cdots \\ = \zeta^{(2m)}(t_{2m}) = \zeta^{(2m+1)}(t_{2m+1}) = 0 \end{aligned} \tag{18}$$

where $t_j \in \{0,1\}$ for all j, $\sum_j t_j = m+1$ and $t_{2m+1-j} = 1 - t_j$. We call this last condition antisymmetry and it guarantees that the covariance kernel (i.e., Green's function) will be positive definite (see [2]).

For a particular antisymmetric choice of $\{t_0, t_1, \ldots, t_{2m+1}\}$ we will call the associated centered Gaussian process a generalized integrated Brownian motion and denote it by $X_{\{t_0,\ldots,t_m\}}(t)$. Notice that we do not need to write the indices t_j for $j > m$ since these are determined by the antisymmetry condition. For example, if $t_0 = t_1 = \cdots = t_m = 0$ then the process is given by (15). If $t_0 = t_2 = \cdots = t_{2m} = 0$ then the process is called an Euler-integrated Brownian motion since the covariance kernel is just the difference of two Euler polynomials. The covariance operator of Euler integrated Brownian motion has the eigenvalues exactly equal to $b_n = ((n-1/2)\pi)^{-2m-2}$ (see [2]).

It will be convenient to define

$$\{n_0, n_1, \ldots, n_m\} = \{i \,:\, t_i = 0\} \quad \text{where} \quad n_0 < n_1 < \cdots < n_m \tag{19}$$

and

$$\{l_0, l_1, \ldots, l_m\} = \{i \,:\, t_i = 1\} \quad \text{where} \quad l_0 < l_1 < \cdots < l_m. \tag{20}$$

In [2] it is shown that as $\varepsilon \to 0$

$$P\left(\|X_{\{t_0,\ldots,t_m\}}\| \le \varepsilon\right) \sim C\varepsilon^{\frac{1}{2m+1}(1-k_0(2m+2))} \exp\{-D_m \varepsilon^{-\frac{2}{2m+1}}\} \tag{21}$$

where

$$D_m = \frac{2m+1}{2}\left((2m+2)\sin\frac{\pi}{2m+2}\right)^{-\frac{2m+2}{2m+1}}, \tag{22}$$

C is a positive constant, and k_0 is an integer. Moreover, in the special case of Euler integrated Brownian motion

$$P\left(\|X_{\{0,1,0,1,\ldots\}}\| \le \varepsilon\right) \sim C_m \varepsilon^{\frac{1}{2m+1}} \exp\{-D_m \varepsilon^{-\frac{2}{2m+1}}\} \quad \text{as} \quad \varepsilon \to 0$$

where

$$C_m = 2^{(m+1)/2}\left(\frac{2m+2}{(2m+1)\pi}\right)^{1/2}\left[(2m+2)\sin\frac{\pi}{2m+2}\right]^{(m+1)/(2m+1)}. \tag{23}$$

We now state

Theorem 4. *Suppose $X_{\{t_0,\ldots,t_m\}}(t)$ is any generalized m-times integrated Brownian motion, i.e., a centered Gaussian process whose covariance operator has eigenvalues and eigenfunctions satisfying the Sturm-Liouville problem (16),(18). Then as $\varepsilon \to 0$*

$$P\left(\|X_{\{t_0,\ldots,t_m\}}\| \le \varepsilon\right) \sim C_{\{t_0,\ldots,t_m\}} \varepsilon^{\frac{1}{2m+2}} \exp\{-D_m \varepsilon^{-\frac{2}{2m+1}}\}$$

where

$$C_{\{t_0,\dots,t_m\}} = \frac{(m+1)^{(m+1)/2}}{|\det(U_{\{t_0,\dots,t_m\}})|}\, C_m,$$

D_m and C_m are defined in (22) and (23), respectively, and

$$U_{\{t_0,\dots,t_m\}} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_{n_0} & \omega_{n_1} & \cdots & \omega_{n_m} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_{n_0}^m & \omega_{n_1}^m & \cdots & \omega_{n_m}^m \end{pmatrix},$$

where $\omega_j = \exp\left(\frac{j\pi}{m+1} i\right)$, and the n_j are given by (19).

Proof. Let us fix an antisymmetric sequence $t_0, t_1, \dots, t_{2m+1}$ corresponding to the boundary condition (18) and denote $X(t) = X_{\{t_0,t_1,\dots,t_m\}}(t)$ the corresponding generalized m-times integrated Brownian motion.

Let $\rho = a^{-1}$ and consider the matrix

$$M(\rho) = (\omega_j^k e^{t_k \alpha_j})_{k,j}, \quad \text{where} \quad \alpha_j = \rho^{1/(2m+2)} i\omega_j, \text{ and } \omega_j = \exp\left(\frac{j\pi}{m+1} i\right).$$

For future developments it will be appropriate to consider the matrix $M(\rho)$ as a function of a complex variable. $M(\rho)$ depends on $\rho \in \mathbb{C}$ as $\rho^{1/(2m+2)}$ so that there are $2m+2$ different choices. However, the symmetries in M guarantee the value of $M(\rho)$ does not depend on this choice as long as we keep the same choice for each element of the matrix. For simplicity take here and in what follows $\rho^{1/(2m+2)}$ lying in the complex sector $-\pi/(2m+2) \le \arg(\rho) < \pi/(2m+2)$. Thus, we will take

$$f(\rho) = \det(M(\rho)), \tag{24}$$

and f is an entire function.

Now, let $a_1 > a_2 > \cdots$ be the eigenvalues corresponding to the covariance operator $\mathcal{A}$ of $X_{\{t_0,t_1,\ldots,t_m\}}(t)$. It is not hard to check that a_n is an eigenvalue of $\mathcal{A}$ if and only if $\{a_n^{-1}\}$ is a root of $f(\rho)$. The goal now is to find an entire function $g(\rho)$ that satisfies the conditions of Theorem 3 and has as its roots the reciprocals of the eigenvalues b_n of the covariance operator $\mathcal{B}$ of Euler-integrated Brownian motion. To carry this out we will first understand the large ρ behavior of f.

Multiply the last m columns of M by e^{α_1}, e^{α_2}, ..., e^{α_m} respectively, use $\alpha_j = -\alpha_{m+1+j}$ and some appropriate row permutations to obtain the following matrix:

$$N = \begin{pmatrix} \omega_0^{n_0} & \cdots & \omega_m^{n_0} & \omega_{m+1}^{n_0} & \omega_{m+2}^{n_0}e^{\alpha_1} & \cdots & \omega_{2m+1}^{n_0}e^{\alpha_m} \\ \omega_0^{n_1} & \cdots & \omega_m^{n_1} & \omega_{m+1}^{n_1} & \omega_{m+2}^{n_1}e^{\alpha_1} & \cdots & \omega_{2m+1}^{n_1}e^{\alpha_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_0^{n_m} & \cdots & \omega_m^{n_m} & \omega_{m+1}^{n_m} & \omega_{m+2}^{n_m}e^{\alpha_1} & \cdots & \omega_{2m+1}^{n_m}e^{\alpha_m} \\ \omega_0^{l_0}e^{\alpha_0} & \cdots & \omega_m^{l_0}e^{\alpha_m} & \omega_{m+1}^{l_0}e^{-\alpha_0} & \omega_{m+2}^{l_0} & \cdots & \omega_{2m+1}^{l_0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_0^{l_m}e^{\alpha_0} & \cdots & \omega_m^{l_m}e^{\alpha_m} & \omega_{m+1}^{l_m}e^{-\alpha_0} & \omega_{m+2}^{l_m} & \cdots & \omega_{2m+1}^{2m+1} \end{pmatrix}$$

where n_j and l_j are given by (19) and (20). The antisymmetry implies the number of 0's and 1's are the same. Note that

$$f(\rho) = e^{\alpha_{m+1}} \cdots e^{\alpha_{2m+1}} \det N(\rho).$$

Further notice for $1 \leq j \leq m$, $\mathrm{Re}(\alpha_j) \leq -\mathrm{Re}(\rho^{1/(2m+2)}) \sin(\pi/(2m+2)) < 0$ and this implies that

$$|e^{\alpha_j}| \leq \exp\left(-\mathrm{Re}(\rho^{1/(2m+2)}) \sin\left(\frac{\pi}{2m+2}\right)\right) \to 0 \text{ as } |\rho| \to \infty.$$

Therefore

$$|\det N - \det N_0| \leq C \exp\left(-\mathrm{Re}(\rho^{1/(2m+2)}) \sin\left(\frac{\pi}{2m+2}\right)\right) \tag{25}$$

where C is a constant depending only on m, and N_0 is the matrix obtained from N by replacing all the entries containing e^{α_j}, $1 \le j \le m$, with 0. That is,

$$N_0 = \begin{pmatrix} \omega_0^{n_0} & \omega_1^{n_0} & \cdots & \omega_m^{n_0} & \omega_{m+1}^{n_0} & 0 & \cdots & 0 \\ \omega_0^{n_1} & \omega_1^{n_1} & \cdots & \omega_m^{n_1} & \omega_{m+1}^{n_1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_0^{n_m} & \omega_1^{n_m} & \cdots & \omega_m^{n_m} & \omega_{m+1}^{n_m} & 0 & \cdots & 0 \\ \omega_0^{l_0} e^{\alpha_0} & 0 & \cdots & 0 & \omega_{m+1}^{l_0} e^{-\alpha_0} & \omega_{m+2}^{l_0} & \cdots & \omega_{2m+1}^{l_0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_0^{l_m} e^{\alpha_0} & 0 & \cdots & 0 & \omega_{m+1}^{l_m} e^{-\alpha_0} & \omega_{m+2}^{l_m} & \cdots & \omega_{2m+1}^{l_m} \end{pmatrix}.$$

It is now easy to see (for details see [2]) that

$$\det N_0 = 2 \det(U) \det(V) \cos(\rho^{1/(2m+2)} \omega_0),$$

where

$$U = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_{n_0} & \omega_{n_1} & \cdots & \omega_{n_m} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_{n_0}^m & \omega_{n_1}^m & \cdots & \omega_{n_m}^m \end{pmatrix}, \quad V = \begin{pmatrix} \omega_{l_0}^{m+1} & \omega_{l_1}^{m+1} & \cdots & \omega_{l_m}^{m+1} \\ \omega_{l_0}^{m+2} & \omega_{l_1}^{m+2} & \cdots & \omega_{l_m}^{m+2} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_{l_0}^{2m+1} & \omega_{l_1}^{2m+1} & \cdots & \omega_{l_m}^{2m+1} \end{pmatrix},$$

Thus,

$$|f(\rho) - 2\det(U)\det(V) \cos(\rho^{1/(2m+2)} \omega_0) e^{\alpha_{m+1} + \cdots + \alpha_{2m+1}}| \\ \le C \exp\left(\mathrm{Re}(\alpha_{m+1} + \cdots + \alpha_{2m+1}) - \mathrm{Re}(\rho^{1/(2m+2)}) \sin\left(\frac{\pi}{2m+2}\right)\right)$$

and, consequently,

$$(26) \quad |f(\rho) - 2^{m+1}\det(U)\det(V) \prod_{j=0}^{m} \cos(\rho^{1/(2m+2)} \omega_j)| \\ \le C' \exp\left(\sum_{j=0}^{2m+1} (\mathrm{Re}(\alpha_j)^+) - |\rho^{1/(2m+2)}| \cos(\frac{\pi}{2m+2}) \sin\left(\frac{\pi}{2m+2}\right)\right),$$

where C' is a constant. Denote

$$g(\rho) = 2^{m+1}\det(U)\det(V)\prod_{j=0}^{m}\cos(\rho^{1/(2m+2)}\omega_j).$$

Similarly as in (24) the value of $g(\rho)$ does not depend on the choice of the root and $g(\rho)$ is an entire function. A simple calculation shows $|\det(U)| = |\det(V)|$ from which it follows that

$$|g(\rho)| = 2^{m+1}|\det(U)|^2\prod_{j=0}^{m}|\cos(\rho^{1/(2m+2)}\omega_j)|. \tag{27}$$

It follows from the properties of cosine that the roots of $g(\rho)$ have the form $\nu_n = ((n-1/2)\pi)^{2m+2}$ for $n = 1, 2, 3 \ldots.$ and $b_n = \nu_n^{-1}$ are exactly the eigenvalues of Euler integrated Brownian motion. (See [2]).

Finally if $|\rho| = (k\pi)^{2m+2}$

$$|g(\rho)| \geq C'' \exp\left(\sum_{j=0}^{2m+1}(\mathrm{Re}(\alpha_j)^+)\right). \tag{28}$$

Combining (26) and (28) we get the condition (4) using $r_n = (n\pi)^{2m+2}$.

Notice that

$$\frac{|f(0)|}{|g(0)|} = \frac{(2m+2)^{m+1}}{2^{m+1}|\det U|^2}$$

and conclude as $\varepsilon \to 0$

$$P(\|X_{\{t_0,t_1,\ldots,t_m\}}\| < \varepsilon) \sim \frac{(m+1)^{(m+1)/2}}{|\det U|}\,P(\|X_{\{0,1,0,1,\ldots\}}\| < \varepsilon)$$

□

Remark 2. It is an easy observation that the constants

$$C_{\{0,0,\ldots,0\}} \geq C_{\{t_0,t_1,\ldots,t_m\}} \geq C_{\{0,1,0,1,\ldots\}}$$

This means that the Euler-integrated Brownian motions have the smallest L^2 small ball probabilities, the usual integrated Brownian motions have the largest L^2 small ball probabilities, and the other general integrated Brownian motions have their small ball probabilities somewhere in between.

Indeed, Hadamard [3] showed, for $(m+1)\times(m+1)$ matrices U with complex entries a_{jk} with $|a_{jk}| \le 1$, $\det(U) \le (m+1)^{(m+1)/2}$ with equality achieved when U is a Vandermonde matrix of $(m+1)$th roots of unity. Notice $\det(U_{\{0,1,0,1,\dots\}}) = (m+1)^{(m+1)/2}$ and thus has the largest possible determinant. Therefore $C_{\{t_0,t_1,\dots,t_m\}} \ge C_{\{0,1,0,1,\dots\}}$ with equality attained only for $\{t_0,t_1,\dots,t_m\} = \{0,1,0,1,\dots\}$ or $\{t_0,t_1,\dots,t_m\} = \{1,0,1,0,\dots\}$.

The antisymmetry assumption provides the following restriction on the set $S = \{n_0,n_1,\dots,n_m\}$: the set S is comprised of exactly one element from each of the pairs $\{0,2m+1\},\{1,2m\},\dots,\{m,m+1\}$. Denote by e_1 either ω_0 or ω_{2m+1} depending on which index was taken from the first pair, similarly define e_2,e_3, etc. Thus, the Vandermonde determinant

$$|\det U_{\{t_0,t_1,\dots,t_m\}}| = \prod_{1\le i<j\le m+1} |e_j - e_i|.$$

In particular, for the usual integrated Brownian motion $e'_1 = \omega_0, \dots, e'_{m+1} = \omega_m$, we have

$$|\det U_{\{0,0,\dots,0\}}| = 2^{\frac{m(m+1)}{2}} \prod_{k=1}^{m} \left(\sin(\frac{k\pi}{2m+2})\right)^{m+1-k}.$$

Symmetry and an application of the law of cosines implies for any $1 \le i < j \le m+1$, $|e'_i - e'_j| \le |e_i - e_j|$. It follows that

$$|\det U_{\{t_0,t_1,\dots,t_m\}}| \ge |\det U_{\{0,0,\dots,0\}}|$$

where equality is attained if and only if $\{t_0, t_1, \ldots, t_m\} = \{0, \ldots, 0\}$ or $\{t_0, t_1, \ldots, t_m\} = \{1, \ldots, 1\}$.

The above result suggests a possible stochastic dominance between these processes: stochastically, the L^2-norm is the largest in the Euler case, and the smallest in the usual case. It would be interesting to have this proved or disproved.

Remark 3. We state the results explicitly in the case of $m = 1$- and 2-times usual integrated Brownian motions, respectively: as $\varepsilon \to 0$

$$P\left(\|X_{\{0,0\}}\| \le \varepsilon\right) \sim \frac{8\sqrt{2}}{\sqrt{3\pi}}\,\varepsilon^{1/3}\exp\{-\frac{3}{8}\varepsilon^{-2/3}\},$$

and

$$P\left(\|X_{\{0,0,0\}}\| \le \varepsilon\right) \sim \frac{36(3^{1/10})}{\sqrt{5\pi}}\,\varepsilon^{1/5}\exp\{-\frac{5}{6(3^{1/5})}\varepsilon^{-2/5}\}.$$

6 Acknowledgements

The authors would like to thank Davar Khoshnevisan and Wenbo Li for their comments and the anonymous referee for the suggestions in simplifying the presentation.

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