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A Characterization of Random Bloch Functions

Fuchang Gao

Department of Mathematics, University of Idaho, Moscow, Idaho 83844-1103
E-mail: fuchang@uidaho.edu

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In this paper, we introduce a necessary and sufficient condition on the complex sequence $\{a_n\}$, $\sum|a_n|^2 < \infty$, so that $\sum_{n=1}^{\infty} \pm a_n z^n$ represents a Bloch function for almost all choices of signs "$\pm$," answering a question left open by J. M. Anderson et al. (1974, *J. Reine Agnew. Math.* **270,** 12–37). © 2000 Academic Press

INTRODUCTION

A Bloch function is an analytic function $f(z)$ in the unit disk $D = \{z : |z| < 1\}$, such that

$$\sup_{z \in D} (1 - |z|^2)|f'(z)| < \infty.$$

When equipped with the norm

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z \in D} (1 - |z|^2)|f'(z)|,$$

the set of all bloch functions forms a Banach space, called the Bloch space.

In this note, we study the random power series

$$f_\omega(z) = \sum_{n=0}^{\infty} a_n \varepsilon_n(\omega) z^n,$$

where $\{\varepsilon_n(\omega)\}$ is a Rademacher sequence; that is, $\varepsilon_n = \pm 1$. In particular, we will consider the following problem raised by Anderson [2]:

Problem. Find a necessary and sufficient condition on $\{a_n\}$, such that for Rademacher sequence $\{\varepsilon_n(\omega)\}$, the series

$$f_\omega(z) = \sum_{n=0}^{\infty} a_n \varepsilon_n(\omega) z^n$$

represents a Bloch function almost surely.

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For history and related research, see, e.g., [2–4].

The study of random series dates back at least to Paley and Zygmund (1930). For a long time, a major question was characterizing the a.s. convergence of the random Fourier series

$$\sum_{n=0}^{\infty} a_n \varepsilon_n e^{ni\theta},$$

where $\{a_n\}$ is a sequence of numbers satisfying $\sum_{n=0}^{\infty}|a_n|^2 < \infty$. This question was completely solved by Marcus and Pisier [6]. Their result will be adapted in this paper to produce the proof of the sufficient part of the following theorem.

THEOREM 1. *If $\{\varepsilon_n\}$ is a Rademacher sequence, then the random power series*

$$f_\omega(z) = \sum_{n=0}^{\infty} a_n \varepsilon_n(\omega) z^n$$

is a Bloch function almost surely if and only if

$$\int_0^\infty \overline{d_n}(e^{-t^2})\, dt = O(n),$$

where $\overline{d_n}$ is the non-decreasing rearrangement of

$$d_n(t) = \sqrt{\sum_{k=1}^{n} k^2 |a_k|^2 |e^{2\pi k t i} - 1|^2}\,.$$

Here and throughout this note, the *non-decreasing rearrangement* of a (Lebesgue) m-measurable function $h(t)$ on $[0,1]$ is defined by

$$\overline{h}(s) = \sup\{y : m(\{t : h(t) < y\}) < s\}.$$

MARCUS AND PISIER

In this section, we introduce a result of Marcus and Pisier [6]. For notational simplicity, we define $\overline{\rho}(t)$ to be the non-decreasing rearrangement of

$$\rho(t) = \left(\sum_{n=0}^{\infty} |a_n|^2 |e^{2\pi n t i} - 1|^2\right)^{1/2},$$

and denote

$$I := \int_0^1 \frac{\bar{\rho}(t)}{t\sqrt{-\log t}}\, dt.$$

The following result can be found in [6, Theorem 1.4, p. 11].

PROPOSITION 1 (Marcus and Pisier). *Let $\{\xi_n\}$ be a sequence of independent, symmetric random variables. Then there exists a constant K, such that*

$$\frac{1}{K}\left(\inf_n E|\xi_n|\right)\left[\sqrt{\sum_{n=0}^{\infty}|a_n|^2} + I\right]$$

$$\leq EZ \leq K\sqrt{\sup_n |E|\xi_n|^2}\left[\sqrt{\sum_{n=0}^{\infty}|a_n|^2} + I\right],$$

where

$$Z := \sup_{0\leq\theta<2\pi}\left|\sum_{n=0}^{\infty} a_n e^{n\theta i}\xi_n(\omega)\right|.$$

For our purposes, we need to improve the right inequality to the following

PROPOSITION 2. *There exists a constant C, such that*

$$\left\|\sup_{0\leq\theta<2\pi}\left|\sum_{n=0}^{\infty} a_n e^{n\theta i}\varepsilon_n\right|\right\|_{\psi_2} \leq C\left[\sqrt{\sum_{n=0}^{\infty}|a_n|^2} + I\right],$$

where the Orlicz norm $\|\cdot\|_{\psi_2}$ is defined by the equation

$$\|x\|_{\psi_2} := \inf\left\{c > 0 : E\exp\left(\frac{|x|^2}{c^2}\right) = 2\right\}.$$

To prove Proposition 2, we need two lemmas. Lemma 1 [4, Theorem 2.1, p. 43] is called the Maurey–Pisier concentration inequality; Lemma 2 [5, p. 97], is a consequence of the contraction principle (see Lemma 3 in the next section).

LEMMA 1. *Let $\{X_t\}_{t\in T}$ be a centered Gaussian process with sample paths bounded a.s. Let $\sigma := \sup_{t\in T} EX_t^2$. Then*

$$P\left\{\left|\sup_{t\in T} X_t - E\sup_{t\in T} X_t\right| > \lambda\right\} \leq 2\exp\left\{-\frac{\lambda^2}{2\sigma^2}\right\}.$$

LEMMA 2. *If $\{g_i(\omega)\}$ is a sequence of i.i.d. standard normal random variables, then*

$$\left\| \sup_{0\le\theta<2\pi} \left| \sum_{n=0}^{\infty} a_n e^{n\theta i} \varepsilon_n \right| \right\|_{\psi_2} \le \left\| \sqrt{\frac{\pi}{2}} \sup_{0\le\theta<2\pi} \left| \sum_{n=0}^{\infty} a_n e^{n\theta i} g_n(\omega) \right| \right\|_{\psi_2}.$$

Proof. Let $\{g_i(\omega)\}$ be a sequence of i.i.d. standard normal random variables. Denote

$$Y_g := \sqrt{\frac{\pi}{2}} \sup_{0\le\theta<2\pi} \sum_{n=0}^{\infty} a_n e^{n\theta i} g_n(\omega)$$

and

$$Z_g := \sqrt{\frac{\pi}{2}} \sup_{0\le\theta<2\pi} \left| \sum_{n=0}^{\infty} a_n e^{n\theta i} g_n(\omega) \right|.$$

By the symmetry of Gaussian variables, we have

$$P\{Z_g > \lambda\} \le 2P\{Y_g > \lambda\}.$$

Using this inequality and then applying Lemma 1 to Y_g, we obtain

$$\|Z_g\|_{\psi_2} \le 2\|Y_g\|_{\psi_2} \le C\left(EY_g + \sqrt{\sum_{n=0}^{\infty} |a_n|^2} \right) \le C\left(EZ_g + \sqrt{\sum_{n=0}^{\infty} |a_n|^2} \right)$$

for some constant C. On the other hand, by applying Proposition 1 to Z_g, we have

$$EZ_g \le K\left[\sqrt{\sum_{n=0}^{\infty} |a_k|^2} + I \right]$$

for some constant K. The proposition follows by invoking Lemma 2. ∎

PROOF OF THEOREM 1

We will need the following contraction principle [5, Theorem 4.4, p. 95].

LEMMA 3. *Let $F\colon \mathbf{R}_+ \to \mathbf{R}_+$ be convex. For any finite sequence (x_k) is a Banach space B and any real numbers (α_k) such that $|\alpha_k| \le 1$ for every k, we have*

$$EF\left(\left\| \sum_k \alpha_k \varepsilon_k x_k \right\| \right) \le EF\left(\left\| \sum_k \varepsilon_k x_k \right\| \right).$$

We start with the following identity. For $z = re^{i\theta}$,

$$\begin{aligned}(1-|z|)|f'_\omega(z)| &= (1-|z|)\left|\sum_{n=1}^{\infty} na_n z^{n-1}\varepsilon_n\right| \\ &= (1-r)\left|\sum_{n=1}^{\infty} nr^{n-1}a_n e^{ni\theta}\varepsilon_n\right| \\ &= \left|\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} ka_k e^{ki\theta}\varepsilon_k\right) r^{n-1}(1-r)^2\right|.\end{aligned}$$

(i) Suppose

$$\int_0^{\infty} \overline{d_n}(e^{-t^2})\,dt = O(n).$$

By changing variable, this is equivalent to

$$\int_0^1 \frac{\overline{d_n}(t)}{t\sqrt{-\log t}}\,dt = O(n).$$

Applying Proposition 2 to the random series $\sum_{k=0}^{n} ka_k e^{k\theta i}\varepsilon_k$, we have

$$\left\|\sup_{0\le\theta<2\pi}\left|\sum_{k=0}^{n} ka_k e^{k\theta i}\varepsilon_k\right|\right\|_{\psi_2} \le C\left[\sqrt{\sum_{k=0}^{n}|ka_k|^2} + \int_0^{\infty}\frac{\overline{d_n}(t)}{t\sqrt{-\log t}}\,dt\right].$$

By Chebyshev's inequality, we deduce that

$$\begin{aligned}\sup_{0\le\theta<2\pi}\left|\sum_{k=1}^{n} ka_k e^{ki\theta}\varepsilon_k\right| &\le n + C\left(\sqrt{\sum_{k=1}^{n}|ka_k|^2} + \int_0^{\infty}\frac{\overline{d_n}(t)}{t\sqrt{-\log t}}\,dt\right) \\ &\le n + Cn\sqrt{\sum_{k=1}^{n}|a_k|^2} + C\int_0^{\infty}\frac{\overline{d_n}(t)}{t\sqrt{-\log t}}\,dt \\ &\le C'n\end{aligned}$$

except on a set with probability less than e^{-n}. (The purpose of Proposition 2 is to produce this quantity.) Thus, with probability more than $1 - \sum_{n=m}^{\infty} e^{-n}$, we have

$$\begin{aligned}
&\sup_{z\in D}(1-|z|)|f'_\omega(z)| \\
&\quad = \sup_{0<r<1}\ \sup_{0\le\theta\le 2\pi}\left|\sum_{n=1}^{\infty} r^{n-1}(1-r)^2 \sum_{k=1}^{n} ka_k e^{ki\theta}\varepsilon_k\right| \\
&\quad \le C_m + \sup_{0<r<1}\sum_{n=m}^{\infty} r^{n-1}(1-r)^2 \sup_{0\le\theta\le 2\pi}\left|\sum_{k=1}^{n} ka_k e^{ki\theta}\varepsilon_k\right| \\
&\quad \le C_m + \sup_{0<r<1}\sum_{n=m}^{\infty} r^{n-1}(1-r)^2 C'n \\
&\quad \le C_m + C' < \infty,
\end{aligned}$$

where C_m is a constant depending on m. This implies $f_\omega(z)$ is a Bloch function almost surely.

(ii) Suppose $f_\omega(z)$ is a Bloch function almost surely. Then the sub-Gaussian process $f'_w(z)$ satisfies

$$E\sup_{z\in D}(1-|z|)|f'_\omega(z)| < \infty.$$

By changing variable, and applying the left inequality of Proposition 1 to the series $\sum_{k=1}^{n} ka_k e^{ki\theta}\varepsilon_k(\omega)$, we have

$$\begin{aligned}
\int_0^\infty \overline{d_n}(e^{-t^2})\,dt &= 2\int_0^1 \frac{\overline{d_n}(t)}{t\sqrt{-\log t}}\,dt \\
&\le 2KE\sup_\theta\left|\sum_{k=1}^{n} ka_k e^{ki\theta}\varepsilon_k(\omega)\right|.
\end{aligned}$$

Consider

$$\frac{1}{n}E\sup_\theta\left|\sum_{k=1}^{n} ka_k e^{ki\theta}\varepsilon_k(\omega)\right|.$$

Because, for $k \le n$, $(1 - \frac{1}{n})^k \ge \frac{1}{e}$, by the contraction principle (Lemma 3),

$$\begin{aligned}
\frac{1}{n} E \sup_{\theta} \left| \sum_{k=1}^{n} k a_k e^{ki\theta} \varepsilon_k(\omega) \right| &\le eE \sup_{\theta} \left| \sum_{k=1}^{n} k a_k e^{ki\theta} \frac{1}{n} \left(1 - \frac{1}{n}\right)^k \varepsilon_k(\omega) \right| \\
&\le eE \sup_{\theta} \left| \sum_{k=1}^{\infty} k a_k e^{ki\theta} \frac{1}{n} \left(1 - \frac{1}{n}\right)^k \varepsilon_k(\omega) \right| \\
&\le eE \sup_{0<r<1} \sup_{\theta} \left| \sum_{k=1}^{\infty} k a_k (1-r) r^k e^{ki\theta} \varepsilon_k(\omega) \right| \\
&= eE \sup_{z \in D} (1 - |z|) \left| \sum_{k=1}^{\infty} k a_k z^k \varepsilon_k(\omega) \right| \\
&= eE \sup_{z \in D} (1 - |z|) |f'_\omega(z)| \\
&< \infty,
\end{aligned}$$

which implies that

$$\int_0^\infty \overline{d_n}(e^{-t^2})\, dt = O(n).$$

COROLLARY 1 (see [2]). *If*

$$\sqrt{\sum_{k=1}^{n} |a_k|^2 k^2} = O\left(\frac{n}{\sqrt{\log n}}\right),$$

then $\sum_{n=0}^{\infty} a_n \varepsilon_n z^n$ *represents a Bloch function almost surely.*

Proof.

$$\begin{aligned}
\int_0^\infty \overline{d_n}(e^{-t^2})\, dt &\le \int_0^\infty \sqrt{\sum_{k=1}^{n} k^2 |a_k|^2 |\exp(2\pi k e^{-t^2} i) - 1|^2}\, dt \\
&\le 2 \int_0^{\sqrt{\log n}} \sqrt{\sum_{k=1}^{n} k^2 |a_k|^2}\, dt \\
&\quad + 8\pi^2 \int_{\sqrt{\log n}}^{\infty} \sqrt{\sum_{k=1}^{n} k^4 |a_k|^2}\, e^{-t^2}\, dt \\
&\le 2\sqrt{\log n} \cdot \sqrt{\sum_{k=1}^{n} k^2 |a_k|^2} + 8\pi^2 \sqrt{\sum_{k=1}^{n} k^2 |a_k|^2} \\
&= O(n).
\end{aligned}$$

The corollary then follows from Theorem 1. ∎

Remark. (i) The readers who are familiar with Marcus and Pisier's proof of Proposition 1 (the idea of replacing a symmetric random variable ξ_n by an identically distributed random variable $\xi_n \varepsilon_n$) should have noticed that Theorem 1 remains valid if ε_n's are replaced by the ξ_n's in Proposition 1. (ii) Anderson also asked the question of characterizing random BMO functions, to which Duren [4] had a very sharp sufficient condition. We note that Duren's sufficient condition can be replaced by a sharper Maurey–Pisier type condition. However, the technique that we used in this paper seems not to work in finding the necessary condition.

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