A Characterization of Random Bloch Functions

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Abstract

In this paper, we introduce a necessary and sufficient condition on the complex sequence $\{a_n\}$, $\sum |a_n|^2 < \infty$, so that $\sum_{n=1}^{\infty} \pm a_n z^n$ represents a Bloch function for almost all choices of signs "±", answering a question left open in [2].

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Introduction

A Bloch function is an analytic function $f(z)$ in the unit disk $D = \{z : |z| < 1\}$, such that

$$\sup_{z \in D} (1 - |z|^2)|f'(z)| < \infty.$$

When equipped with the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in D} (1 - |z|^2)|f'(z)|,$$

the set of all Bloch functions forms a Banach space, called the Bloch space.

In this note, we study the random power series

$$f_\omega(z) = \sum_{n=0}^{\infty} a_n \varepsilon_n(\omega) z^n$$

where $\{\varepsilon_n(\omega)\}$ is a Rademacher sequence, that is $\varepsilon_n = \pm 1$. In particular, we will consider the following problem raised by Anderson in [1]:

Problem *Find a necessary and sufficient condition on $\{a_n\}$, such that for Rademacher sequence $\{\varepsilon_n(\omega)\}$, the series*

$$f_\omega(z) = \sum_{n=0}^{\infty} a_n \varepsilon_n(\omega) z^n$$

represents a Bloch function almost surely.

For the history and the related research, see e.g. [2], [3] and [1].

The study of random series dates back at least to Paley and Zygmund (1930). For a long time, a major question was to characterize the a.s. convergence of the random Fourier series

$$\sum_{n=0}^{\infty} a_n \varepsilon_n e^{ni\theta},$$

where $\{a_n\}$ is a sequence of numbers satisfying $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. This question was completely solved by Marcus and Pisier ([4]). Their result will be adapted in this paper to produce the proof of the sufficient part of the following theorem.

Theorem 1 *If $\{\varepsilon_n\}$ is a Rademacher sequence, then the random power series*

$$f_\omega(z) = \sum_{n=0}^{\infty} a_n \varepsilon_n(\omega) z^n$$

is a Bloch function almost surely if and only if

$$\int_0^{\infty} \overline{d_n}(e^{-t^2})dt = O(n),$$

where $\overline{d_n}$ is the non-decreasing rearrangement of

$$d_n(t) = \sqrt{\sum_{k=1}^{n} k^2 |a_k|^2 |e^{2\pi k t i} - 1|^2}.$$

Here and throughout this note, the *non-decreasing rearrangement* of a (Lebesgue) m-measurable function $h(t)$ on $[0, 1]$ is defined by

$$\overline{h}(s) = \sup\{y : m(\{t : h(t) < y\}) < s\}.$$

Marcus-Pisier

In this section, we introduce a result of Marcus and Pisier [4]. For the notational simplicity, we define $\overline{\rho}(t)$ to be the non-decreasing rearrangement of

$$\rho(t) = \left(\sum_{n=0}^{\infty} |a_n|^2 |e^{2\pi n t i} - 1|^2\right)^{1/2},$$

and denote

$$I := \int_0^1 \frac{\overline{\rho}(t)}{t\sqrt{-\log t}} dt.$$

The following result can be found in [4] (p.11, Th. 1.4).

Proposition 1 *(***Marcus-Pisier***) Let $\{\xi_n\}$ be a sequence of independent, symmetric random variables. Then there exists a constant K, such that*

$$\frac{1}{K}\left(\inf_n E|\xi_n|\right)\left[\sqrt{\sum_{n=0}^{\infty}|a_n|^2}+I\right] \leq EZ \leq K\sqrt{\sup_n E|\xi_n|^2}\left[\sqrt{\sum_{n=0}^{\infty}|a_n|^2}+I\right]$$

where

$$Z := \sup_{0\leq\theta<2\pi}\left|\sum_{n=0}^{\infty}a_n e^{n\theta i}\xi_n(\omega)\right|.$$

For our purpose, we need to improve the right inequality to the following

Proposition 2 *There exists a constant C, such that*

$$\left\|\sup_{0\leq\theta<2\pi}\left|\sum_{n=0}^{\infty}a_n e^{n\theta i}\varepsilon_n\right|\right\|_{\psi_2} \leq C\left[\sqrt{\sum_{n=0}^{\infty}|a_n|^2}+I\right],$$

where the Orlicz norm $\|\cdot\|_{\psi_2}$ is defined by the equation

$$\|x\|_{\psi_2} := \inf\left\{c>0 : E\exp\left(\frac{|x|^2}{c^2}\right)=2\right\}.$$

To prove Proposition 2, we need two lemmas. Lemma 1 ([6], p. 43, Theorem 2.1) is called Maurey-Pisier concentration inequality; Lemma 2 ([5], p.97) is a consequence of the contraction principle (see Lemma 3 in the next section).

Lemma 1 *Let $\{X_t\}_{t\in T}$ be a centered Gaussian processes with sample paths bounded a.s. Let $\sigma := \sup_{t\in T} EX_t^2$. Then*

$$P\left\{\left|\sup_{t\in T}X_t - E\sup_{t\in T}X_t\right| > \lambda\right\} \leq 2\exp\left\{-\frac{\lambda^2}{2\sigma^2}\right\}.$$

Lemma 2 *If $\{g_i(\omega)\}$ is a sequence of i.i.d standard normal random variables, then*

$$\left\|\sup_{0\leq\theta<2\pi}\left|\sum_{n=0}^{\infty}a_n e^{n\theta i}\varepsilon_n\right|\right\|_{\psi_2} \leq \left\|\sqrt{\frac{\pi}{2}}\sup_{0\leq\theta<2\pi}\left|\sum_{n=0}^{\infty}a_n e^{n\theta i}g_n(\omega)\right|\right\|_{\psi_2}.$$

Proof: Let $\{g_i(\omega)\}$ be a sequence of i.i.d standard normal random variables. Denote

$$Y_g := \sqrt{\frac{\pi}{2}}\sup_{0\leq\theta<2\pi}\sum_{n=0}^{\infty}a_n e^{n\theta i}g_n(\omega)$$

and

$$Z_g := \sqrt{\frac{\pi}{2}}\sup_{0\leq\theta<2\pi}\left|\sum_{n=0}^{\infty}a_n e^{n\theta i}g_n(\omega)\right|.$$

By the symmetry of Gaussian variables, we have

$$P\{Z_g > \lambda\} \leq 2P\left\{Y_g > \lambda\right\}.$$

Using this inequality and then applying Lemma 1 to Y_g, we obtain

$$\|Z_g\|_{\psi_2} \leq 2\,\|Y_g\|_{\psi_2} \leq C\left(EY_g + \sqrt{\sum_{n=0}^{\infty}|a_n|^2}\right) \leq C\left(EZ_g + \sqrt{\sum_{n=0}^{\infty}|a_n|^2}\right)$$

for some constant C. On the other hand, by applying Proposition 1 to Z_g, we have

$$EZ_g \;\;\leq\;\; K\left[\sqrt{\sum_{n=0}^{\infty}|a_k|^2} + I\right]$$

for some constant K. The Proposition follows by invoking Lemma 2. □

Proof of Theorem 1

We will need the following contraction principle ([5], p. 95, Theorem 4.4).

Lemma 3 *Let F: $\mathbf{R}_+ \to \mathbf{R}_+$ be convex. For any finite sequence (x_k) in a Banach space B and any real numbers (α_k) such that $|\alpha_k| \leq 1$ for every k, we have*

$$EF\left(\left\|\sum_k \alpha_k\varepsilon_k x_k\right\|\right) \leq EF\left(\left\|\sum_k \varepsilon_k x_k\right\|\right).$$

We start with the following identity. For $z = re^{i\theta}$,

$$\begin{aligned}(1-|z|)\,|f'_\omega(z)| &= (1-|z|)\left|\sum_{n=1}^{\infty} na_n z^{n-1}\varepsilon_n\right| \\ &= (1-r)\left|\sum_{n=1}^{\infty} nr^{n-1}a_n e^{ni\theta}\varepsilon_n\right| \\ &= \left|\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} ka_k e^{ki\theta}\varepsilon_k\right) r^{n-1}(1-r)^2\right|.\end{aligned}$$

(i) Suppose

$$\int_0^{\infty} \overline{d_n}(e^{-t^2})dt = O(n).$$

By changing variable, this is equivalent to

$$\int_0^1 \frac{\overline{d_n}(t)}{t\sqrt{-\log t}}dt = O(n).$$

Applying Proposition 2 to the random series $\sum_{k=0}^{n} ka_k e^{k\theta i}\varepsilon_k$, we have

$$\left\| \sup_{0\le\theta<2\pi} \left| \sum_{k=0}^{n} ka_k e^{k\theta i}\varepsilon_k \right| \right\|_{\psi_2} \le C \left[\sqrt{\sum_{k=0}^{n} |ka_k|^2} + \int_0^\infty \frac{\overline{d_n}(t)}{t\sqrt{-\log t}} dt \right].$$

By Chebyshev's inequality, we deduce that

$$\begin{aligned} \sup_{0\le\theta<2\pi} \left| \sum_{k=1}^{n} ka_k e^{ki\theta}\varepsilon_k \right| &\le n + C \left(\sqrt{\sum_{k=1}^{n} |ka_k|^2} + \int_0^\infty \frac{\overline{d_n}(t)}{t\sqrt{-\log t}} dt \right) \\ &\le n + Cn\sqrt{\sum_{k=1}^{n} |a_k|^2} + C\int_0^\infty \frac{\overline{d_n}(t)}{t\sqrt{-\log t}} dt \\ &\le C'n \end{aligned}$$

except on a set with probability less than e^{-n}. (The purpose of Proposition 2 is to produce this quantity.) Thus, with probability more than $1 - \sum_{n=m}^{\infty} e^{-n}$, we have

$$\begin{aligned} \sup_{z\in D}(1-|z|)\,|f'_\omega(z)| &= \sup_{0<r<1} \sup_{0\le\theta\le 2\pi} \left| \sum_{n=1}^{\infty} r^{n-1}(1-r)^2 \sum_{k=1}^{n} ka_k e^{ki\theta}\varepsilon_k \right| \\ &\le C_m + \sup_{0<r<1} \sum_{n=m}^{\infty} r^{n-1}(1-r)^2 \sup_{0\le\theta\le 2\pi} \left| \sum_{k=1}^{n} ka_k e^{ki\theta}\varepsilon_k \right| \\ &\le C_m + \sup_{0<r<1} \sum_{n=m}^{\infty} r^{n-1}(1-r)^2 C'n \\ &\le C_m + C' < \infty \end{aligned}$$

where C_m is a constant depending on m. This implies $f_\omega(z)$ is a Bloch function almost surely.

(ii) Suppose $f_\omega(z)$ is a Bloch function almost surely. Then

$$\sup_{z\in D}(1-|z|)\,|f'_\omega(z)| < \infty.$$

By changing variable, and applying the left inequality of Proposition 1 to the series $\sum_{k=1}^{n} ka_k e^{ki\theta}\varepsilon_k(\omega)$, we have

$$\begin{aligned} \int_0^\infty \overline{d_n}(e^{-t^2})dt &= 2\int_0^1 \frac{\overline{d_n}(t)}{t\sqrt{-\log t}} dt \\ &\le 2KE\sup_\theta \left| \sum_{k=1}^{n} ka_k e^{ki\theta}\varepsilon_k(\omega) \right|. \end{aligned}$$

Consider

$$\frac{1}{n} E\sup_\theta \left| \sum_{k=1}^{n} ka_k e^{ki\theta}\varepsilon_k(\omega) \right|.$$

Because for $k \leq n$, $\left(1-\frac{1}{n}\right)^k \geq \frac{1}{e}$, by the contraction principle (Lemma 3),

$$\begin{aligned}
\frac{1}{n} E \sup_\theta \left|\sum_{k=1}^n k a_k e^{ki\theta} \varepsilon_k(\omega)\right| &\leq eE \sup_\theta \left|\sum_{k=1}^n k a_k e^{ki\theta} \frac{1}{n}\left(1-\frac{1}{n}\right)^k \varepsilon_k(\omega)\right| \\
&\leq eE \sup_\theta \left|\sum_{k=1}^\infty k a_k e^{ki\theta} \frac{1}{n}\left(1-\frac{1}{n}\right)^k \varepsilon_k(\omega)\right| \\
&\leq eE \sup_{0<r<1} \sup_\theta \left|\sum_{k=1}^\infty k a_k (1-r) r^k e^{ki\theta} \varepsilon_k(\omega)\right| \\
&= eE \sup_{z\in D} (1-|z|) \left|\sum_{k=1}^\infty k a_k z^k \varepsilon_k(\omega)\right| \\
&= e \sup_{z\in D} (1-|z|) \left|f'_\omega(z)\right| \\
&< \infty,
\end{aligned}$$

which implies that

$$\int_0^\infty \overline{d_n}(e^{-t^2}) dt = O(n).$$

Corollary 1 *(see [2]) If*

$$\sqrt{\sum_{k=1}^n |a_k|^2 k^2} = O\left(\frac{n}{\sqrt{\log n}}\right),$$

then $\sum_{n=0}^\infty a_n \varepsilon_n z^n$ *represents a Bloch function almost surely.*

Proof:

$$\begin{aligned}
\int_0^\infty \overline{d_n}(e^{-t^2}) dt &\leq \int_0^\infty \sqrt{\sum_{k=1}^n k^2 |a_k|^2 |\exp(2\pi k e^{-t^2} i) - 1|^2} dt \\
&\leq 2\int_0^{\sqrt{\log n}} \sqrt{\sum_{k=1}^n k^2 |a_k|^2} dt + 8\pi^2 \int_{\sqrt{\log n}}^\infty \sqrt{\sum_{k=1}^n k^4 |a_k|^2 e^{-t^2}} dt \\
&\leq 2\sqrt{\log n} \cdot \sqrt{\sum_{k=1}^n k^2 |a_k|^2} + 8\pi^2 \sqrt{\sum_{k=1}^n k^2 |a_k|^2} \\
&= O(n).
\end{aligned}$$

The Corollary then follows from Theorem 1. □

Remark: (i) The readers who are familiar with Marcus-Pisier's proof of Proposition 1 (the idea of replacing a symmetric random variable ξ_n by an identically distributed random variable $\xi_n \varepsilon_n$) should have noticed that Theorem 1 remains valid if ε_n's are replaced by the ξ_n's in Proposition 1. (ii) Anderson also asked the question of characterizing random BMO functions, to which Duren had a very sharp sufficient condition. We note that Duren's sufficient condition can be replaced by a sharper Maurey-Pisier type condition. However, the technique that we used in this paper seems not to work in finding the necessary condition.

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