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BIPACKING A BIPARTITE GRAPH WITH GIRTH AT LEAST 12

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Abstract. Let G be a bipartite graph with (X, Y) as its bipartition. Let B be a complete bipartite graph with a bipartition (V_1, V_2) such that $X \subseteq V_1$ and $Y \subseteq V_2$. A *bi-packing* of G in B is an injection σ: $V(G) \to V(B)$ such that $\sigma(X) \subseteq V_1$, $\sigma(Y) \subseteq V_2$ and $E(G) \cap E(\sigma(G)) = \emptyset$. In this paper, we show that if G is a bipartite graph of order n with girth at least 12, then there is a complete bipartite graph B of order $n+1$ such that there is a bi-packing of G in B. We conjecture that the same conclusion holds if the girth of G is at least 8.

1. Introduction

For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. In this paper, we denote a bipartite graph G with a given bipartition (X, Y) by $G(X, Y)$, and for a bipartite graph, we always assume that it has been given a bipartition. If H is a subgraph of $G(X, Y)$, then the bipartition of H is given as $(V(H) \cap X, V(H) \cap Y)$. We use B_n to denote a complete bipartite graph of order n. Let $G(X, Y)$ and $H(U, W)$ be two bipartite graphs. Let $B_n(V_1, V_2)$ be such that $U \subseteq V_1$ and $W \subseteq V_2$. A bipacking of G and H in $B_n(V_1, V_2)$ is a bijection σ: $V(G) \to V(B_n)$ such that $\sigma(X) \subseteq V_1$, $\sigma(Y) \subseteq V_2$ and $E(H) \cap E(\sigma(G)) = \emptyset$, where $\sigma(G)$ is the image of G under σ. If additionally $G = H$, we say that there is a bipacking of G in B_n. Fouquet and Wojda [4] showed that for any disconnected forest F of order n, there is a bipacking of F in a B_n. This result was also obtained by Sauer and Wang [7]. Two bipartite graphs $G(X, Y)$ and $H(U, W)$ are compatible if $|X| = |U|$ and $|Y| = |W|$. In [8], we proved the following:

Theorem A ([8]). *Let D and F be two compatible disconnected forests of order n. Suppose that D and F can be partitioned into vertex-disjoint unions of subforests $D = D_1 \cup D_2$ and $F = F_1 \cup F_2$ such that D_i and F_i are compatible for $i = 1, 2$. Then there is a bipacking of D and F in a B_n.*

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In [9], we investigated a bipacking of two compatible bipartite graphs G and H of order n with $e(G) + e(H) \le 2n - 2$, and we showed:

Theorem B ([9]). *Let G and H be two compatible bipartite graphs of order n with $e(G) + e(H) \le 2n - 2$. Suppose that each of G and H does not contain a cycle of length 4. Then there exists a complete bipartite graph B of order $n+1$ such that there is a bipacking of G and H in B unless one of G and H is the union of vertex-disjoint cycles and the other is the union of two vertex-disjoint stars.*

In this paper, we investigate the bipacking of a bipartite graph G with girth at least 12. This work is motivated by a conjecture in [3] and the result in [2]. R. J. Faudree, C. C. Rousseau, R. H. Schelp and S. Schuster conjectured in [3] that if G is a graph of order n with girth at least 5 and maximum degree at most $n-2$, then there is an embedding of G in its complement. S. Brandt proved in [2] that if the girth of G is at least 7, then the conclusion holds. Görlich, Poliśniak, Woźniak and Ziolo provided a simpler proof of this result in [5], whose idea is adopted in our current work. For bipartite graphs, we conjecture the following:

Conjecture C. *If G is a bipartite graph of order n with girth at least 8, then there is a bipacking of G in a complete bipartitite graph of order $n+1$.*

This conjecture holds for trees by Theorem B. Orchel characterized all the trees of order n that do not have bipackings in complete bipartite graphs of order n. There are three types of those trees and we refer readers to [6] for a list of them. In this paper, we will prove the following result:

Theorem D. *If G is a bipartite graph of order n with girth at least 12, then there is a bipacking of G in a complete bipartitite graph of order $n+1$.*

To prove Theorem D, we will prove Theorem E which is stronger than Theorem D. To state Theorem E, we define F_n to be a tree of order n with $n \ge 5$ such that F_n has a path $x_1x_2x_3x_4$ of order 4 and every vertex in $V(F_n) - \{x_1, x_2, x_3, x_4\}$ is an endvertex adjacent to x_4. We use $2K_2$ to denote the graph of order 4 which consists of two independent edges. Let $\mathcal{F}$ be a set of graphs such that a graph H belongs to $\mathcal{F}$ if and only if either H is isomorphic to one of $2K_2$, P_4, P_6 and F_n for some $n \ge 5$ or H has order 2 and each partite in the given bipartition of H is non-empty. Note that F_5 is P_5. A bipacking σ of a bipartite graph G in a complete bipartite is called a fixed-point-free (FPF) bipacking if $\sigma(x) \ne x$ for all $x \in V(G)$. For convenience, we denote the order of a graph G by $|G|$. It is easy to check that each graph H in $\mathcal{F}$ has a bipacking in a $B_{|H|+1}$ but does not have an FPF bipacking in a $B_{|H|+1}$.

Theorem E. *If G is a bipartite graph of order n with girth at least 12, then there is an FPF bipacking of G in a complete bipartitite graph of order $n+1$ if and only if G does not belong to $\mathcal{F}$.*

We discuss only finite simple graphs and use standard terminology and notation from [1] unless indicated otherwise. Here we define some special terminology and notation to be used in this paper. Let G be a graph. Let X be a subset of $V(G)$ or a subgraph of G. We define $G[X]$ to be the subgraph induced by the vertices belonging to X. If Y is a subset of $V(G)$ or a subgraph of G such that X and Y do not have any common vertex, then we define $E(X,Y)$ to be the set of edges between X and Y in G and let $e(X,Y) = |E(X,Y)|$. For a vertex x of G, we define $d(x,X)$ to be the number of neighbors of x in G that are contained in X. Thus $d(x,G)$ is the degree of x in G. For a subset Z of $V(G)$, let $N(Z) = \cup_{z \in Z} N(z)$. We use $|G|$ to denote the order of G.

A *feasible* path of G is an induced path of order 4 in G such that each of its two internal vertices has degree 2 in G. A *feasible* edge of G is an edge xy of G such that $d_G(x) = d_G(y) = 2$.

Note that the girth of a graph without cycles is defined to be infinity ∞.

2. Proof of Theorem E

On the contrary, we suppose that Theorem E fails. Let $G(X_1, X_2)$ be a bipartite graph with the smallest order such that the girth of G is at least 12 and $G \notin \mathcal{F}$ but G does not have an FPF bipacking in a B_{n+1}, where $n = |G|$. Let

$$\begin{aligned} &n_1 = |X_1| \ \text{ and } \ n_2 = |X_2|; \\ &\delta_1 = \min_{x \in X_1} d(x) \ \text{ and } \ \delta_2 = \min_{x \in X_2} d(x); \\ &\Delta_1 = \max_{x \in X_1} d(x) \ \text{ and } \ \Delta_2 = \max_{x \in X_2} d(x). \end{aligned}$$

Clearly, $n = n_1 + n_2$, $\delta(G) = \min\{\delta_1, \delta_2\}$, $n_1 \geq 2$ and $n_2 \geq 2$. Our proof consists of the following lemmas, which will lead to a contradiction.

Lemma 2.1. *Let $k \geq 2$. If $x_1, x_2, \ldots, x_k$ are k distinct endvertices of G with a common neighbor, then $G - \{x_1, x_2, \ldots, x_k\}$ does not have an FPF bipacking in a B_{n-k+1}.*

Proof. If $G - \{x_1, x_2, \ldots, x_k\}$ has an FPF bipacking σ in a B_{n-k+1}, then σ can be extended to an FPF bipacking of G in a B_{n+1} such that $\sigma(x_i) = x_{i+1}$ for all $i \in \{1, \ldots, k\}$ where $x_{k+1} = x_1$, a contradiction. □

Lemma 2.2. *The following two statements hold*:

(a) *There exists no $x \in V(G)$ such that $G - x$ has an FPF bipacking in a B_{n-1}.*
(b) *There exists no $z \in V(G)$ such that $G - z \in \mathcal{F}$.*

Proof. If $G - x$ has an FPF bipacking σ in a B_{n-1} for some $x \in V(G)$, let w be a new vertex not in G and we extend σ with $\sigma(x) = w$. Then σ becomes an FPF bipacking of G in a B_{n+1}, a contradiction. Hence (a) holds.

To see (b), we suppose that $G - z \in \mathcal{F}$ for some $z \in V(G)$. If $|G - z| = 2$, we readily see that G has an FPF bipacking in a B_4. Hence $n \geq 5$. Then we see that $d(z) \leq 1$ since $G \notin \mathcal{F}$ (in particular, $G \ncong P_5$) and $g(G) \geq 8$. Let w be a new vertex not in G. We define an injection $\sigma : V(G) \to V(G) \cup \{w\}$ with $\sigma(x) \neq x$ for all $x \in V(G)$ as follows.

First, assume that $G - z \cong 2K_2$ or P_4. Let x_1x_2 and x_3x_4 be two edges of $G - z$ with $\{x_1, x_3\} \subseteq X_1$ such that $d_{G-z}(x_1) = d_{G-z}(x_4) = 1$. As $d_G(z) \leq 1$ and $G \notin \mathcal{F}$, we may assume that $N_G(z) \subseteq \{x_2\}$ or $N_G(z) \subseteq \{x_3\}$. Say w.l.o.g. that $N_G(z) \subseteq \{x_3\}$ and x_3 and z are not in the same partite of G. Let

$$\sigma(x_1, x_2, x_3, x_4, z) = (x_3, w, x_1, z, x_4).$$

Next, assume that $G - z \cong P_6$. Say $G - z = x_1x_2x_3x_4x_5x_6$. If $N(z) \subseteq \{x_1\}$ or $N(z) \subseteq \{x_6\}$, say w.l.o.g. $N(z) \subseteq \{x_6\}$ and x_6 and z are not in the same partite of G, let

$$\sigma(x_1, x_2, x_3, x_4, x_5, x_6, z) = (x_3, w, x_5, x_2, z, x_4, x_1).$$

If $N(z) \nsubseteq \{x_1, x_6\}$, then $N(z) = \{x_i\}$ for some $i \in \{2, 3, 4, 5\}$. Say w.l.o.g. $N(z) = \{x_i\}$ with $i \in \{4, 5\}$. Let

$$\sigma(x_1, x_2, x_3, x_4, x_5, x_6, z) = (x_3, x_6, x_1, w, z, x_2, x_5) \quad \text{if} \quad i = 4;$$
$$\sigma(x_1, x_2, x_3, x_4, x_5, x_6, z) = (x_3, x_6, x_1, z, w, x_2, x_4) \quad \text{if} \quad i = 5.$$

Finally, assume that $G - z \cong F_{n-1}$ with $n - 1 \geq 5$. Say

$$V(G - z) = \{x_1, x_2, x_3, x_4\} \cup \{a_1, a_2, \ldots, a_k\}$$

such that $x_1x_2x_3x_4$ is a path in G and $N_{G-z}(x_4) = \{x_3, a_1, a_2, \ldots, a_k\}$. Set $A = \{a_1, a_2, \ldots, a_k\}$ and $a_{k+1} = a_1$. Then $zx_4 \notin E$ as $G \notin \mathcal{F}$. If $zx_2 \in E$, then $k \geq 2$ as $G \notin \mathcal{F}$. Thus if $zx_2 \in E$, we see that $G - A \notin \mathcal{F}$ and so $G - A$ has an FPF bipacking in a B_6, contradicting Lemma 2.1. Hence $zx_2 \notin E$. Similarly, if $zx_3 \in E$, then $k = 1$. If $zx_1 \in E$, then $k \geq 2$ as $G \ncong P_6$. If $N(z) \subseteq A$, we may assume that $N(z) \subseteq \{a_k\}$. Let

$$\sigma(x_1, x_2, x_3, x_4, z, a_1, \ldots, a_k) = (a_1, z, a_2, w, x_2, a_3, \ldots, a_k, x_1, x_3) \quad \text{if} \quad zx_1 \in E;$$
$$\sigma(x_1, x_2, x_3, x_4, a_1, z) = (x_3, w, a_1, z, x_1, x_2) \quad \text{if} \quad zx_3 \in E;$$
$$\sigma(x_1, x_2, x_3, x_4, z, a_1, \ldots, a_k) = (x_3, z, x_1, w, x_2, a_2, \ldots, a_{k+1}) \quad \text{if} \quad N(z) \subseteq \{a_k\}.$$

In each of the above situations, we see that σ is an FPF bipacking of G in a B_{n+1}, a contradiction. □

Lemma 2.3. *Let $\{i, j\} = \{1, 2\}$. Let $x \in X_i$, $Y = N_G(x)$ and $H = G - x$. Let σ be an FPF bipacking of H in a $B_n(V_1, V_2)$ with $X_i - \{x\} \subseteq V_i$ and $X_j \subseteq V_j$. Then $V_i - \{x\} \subseteq N_{\sigma(H)}(Y) \cup N_H(\sigma(Y))$. Moreover, there exists a subset $W \subseteq X_j$ such that*

$$|W| = |N_G(x)| \quad \text{and} \quad |N_{G-x}(W)| \geq \frac{1}{2}(n_i - 1).$$

Proof. For convenience, say $i = 1$ and $j = 2$. Assume that there exists $u \in V_1 - \{x\}$ such that $u \notin N_{\sigma(H)}(Y) \cup N_H(\sigma(Y))$. If $\sigma^{-1}(u)$ does not exist, then we obtain an FPF bipacking σ' of G in $B_{n+1}(V_1 \cup \{x\}, V_2)$ with $\sigma'(x) = u$ and $\sigma'(w) = \sigma(w)$ for all $w \in V(G) - \{x\}$, a contradiction. Therefore $\sigma^{-1}(u)$ exists. Let $v = \sigma^{-1}(u)$. Then we obtain an FPF bipacking σ' of G in $B_{n+1}(V_1 \cup \{x\}, V_2)$ with $\sigma'(x) = u$, $\sigma'(v) = x$ and $\sigma'(w) = \sigma(w)$ for all $w \in V(G) - \{x, v\}$, a contradiction. Therefore $V_1 - \{x\} \subseteq N_{\sigma(H)}(Y) \cup N_H(\sigma(Y))$. This implies that $n_1 - 1 \leq |N_{\sigma(H)}(Y)| + |N_H(\sigma(Y))|$. Let $A = \{z \in X_2 | \sigma(z) \in Y\}$. Note that since $|X_2| \leq |V_2| \leq |X_2| + 1$, we see that $|Y| - 1 \leq |A| \leq |Y|$. Then $N_{\sigma(H)}(Y) = \sigma(N_H(A))$ and so $n_1 - 1 \leq |N_H(A)| + |N_H(\sigma(Y))|$. Let $A \subseteq A' \subseteq X_2$ with $|A'| = |Y|$. Then $n_1 - 1 \leq |N_H(A')| + |N_H(\sigma(Y))|$. Thus either $|N_H(A')| \geq (n_1 - 1)/2$ or $|N_H(\sigma(Y))| \geq (n_1 - 1)/2$. This means that the lemma holds with either $W = A'$ or $W = \sigma(Y)$. □

Corollary 2.4. *$\delta(G) > 0$, $n_1 \leq 1 + 2\delta_1\Delta_2$ and $n_2 \leq 1 + 2\delta_2\Delta_1$.*

Proof. For each $x \in V(G)$, we see that $N(x) \neq \emptyset$ by Lemma 2.2 and Lemma 2.3. To see the inequality $n_1 \leq 1 + 2\delta_1\Delta_2$, we choose $x \in X_1$ with $d(x) = \delta_1$. By Lemma 2.3, $(n_1 - 1)/2 \leq \delta_1\Delta_2$, i.e., $n_1 \leq 1 + 2\delta_1\Delta_2$. Similarly, $n_2 \leq 1 + 2\delta_2\Delta_1$. □

Corollary 2.5. *If x is an endvertex of G and y is the neighbor of x, then $d_G(y) \leq 2$.*

Proof. Say $x \in X_1$. By Lemma 2.2, $G - x \notin \mathcal{F}$. Then $G - x$ has an FPF bipacking σ in a $B_n(V_1, V_2)$ with $X_1 - \{x\} \subseteq V_1$ and $X_2 \subseteq V_2$. By Lemma 2.3, we see, with $Y = \{y\}$ and $H = G - x$, that $V_1 - \{x\} \subseteq N_{\sigma(G-x)}(y) \cup N_{G-x}(\sigma(y))$. Then $N_{G-x}(y) \subseteq N_{G-x}(\sigma(y))$. As $g(G) > 4$, it follows that $|N_{G-x}(y)| \leq 1$ and so $d_G(y) \leq 2$. □

Lemma 2.6. *If P is a path of order $t \geq 8$ from x to y, then there is an FPF bipacking τ of P in a B_t such that $\tau(x)\tau(y) \notin E(P)$.*

Proof. Say $P = x_1 y_1 \cdots x_k y_k$ if $t = 2k$ and $P = x_1 y_1 \cdots x_k y_k x_{k+1}$ if $t = 2k+1$. Let $x_{\lceil t/2 \rceil + 1} = x_1$ and $y_0 = y_{\lfloor t/2 \rfloor}$. Let τ be defined as follows:

$$\tau(x_i) = x_{i+1} \text{ for } i \in \{1, 2, \ldots, \lceil t/2 \rceil\} \text{ and } \tau(y_j) = y_{j-1} \text{ for } j \in \{1, 2, \ldots, \lfloor t/2 \rfloor\}.$$

It is easy to see that τ satisfies the requirement. □

Corollary 2.7. *Every bipartite graph $H(V_1, V_2)$ of order $n \geq 8$ with girth at least 8, $\Delta(H) \leq 2$ and $||V_1| - |V_2|| \leq 1$ has an FPF bipacking in a B_n.*

With Corollary 2.7 and Lemma 2.2(a), we obtain:

Corollary 2.8. *There exists no $x \in V(G)$ such that $G - x$ is a linear forest of order at least 8 with $||V(G - x) \cap X_1| - |V(G - x) \cap X_2|| \leq 1$.*

Lemma 2.9. *The graph G does not contain two vertex-disjoint feasible paths.*

Proof. On the contrary, say the lemma fails. Let $P = x_1x_2x_3x_4$ and $Q = y_1y_2y_3y_4$ be two vertex-disjoint feasible paths with $\{x_1, y_1\} \subseteq X_1$. Let $H = G - V(P \cup Q)$. Assume for the moment that $H \notin \mathcal{F}$. Let σ be an FPF bipacking of H in a $B_{n-7}(V_1, V_2)$ with $X_1 - \{x_1, x_3, y_1, y_3\} \subseteq V_1$ and $X_2 - \{x_2, x_4, y_2, y_4\} \subseteq V_2$. We extend σ to an FPF bipacking of G in $B_{n+1}(V_1 \cup \{x_1, x_3, y_1, y_3\}, V_2 \cup \{x_2, x_4, y_2, y_4\})$ by setting

$$\sigma(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (y_3, x_4, x_1, y_2, x_3, y_4, y_1, x_2).$$

This contradicts the assumption on G.

Therefore $H \in \mathcal{F}$. Let w be a new vertex not in G. Since $g(G) \geq 12$, we see that if $|H| = 2$, then $|V(H) \cap X_1| = |V(H) \cap X_2| = 1$ and $e(\{x_1, x_4, y_1, y_4\}, H) + e(H) \leq 3$ and if H is one of P_2, P_4, P_5, P_6 and F_n, then $e(\{x_1, x_4, y_1, y_4\}, H) \leq 2$. Moreover, with Corollary 2.7, we see that if H is $2K_2$, then $e(\{x_1, x_4, y_1, y_4\}, H) \leq 3$. By Corollary 2.7 and Corollary 2.8, we readily see that $|H| \neq 2$ and $H \neq 2K_2$. We shall construct an FPF bipacking σ of G in a B_{n+1}.

First, assume that H is one of P_4, P_5 and P_6. By Corollary 2.8, we see that H contains two distinct vertices v_1 and v_2 such that $d_G(v_1) \geq 3$ and $d_G(v_2) \geq 3$ and each endvertex of H is still an endvertex of G. By Corollary 2.5 and as $g(G) \geq 8$, it follows that H is a path $a_1a_2a_3a_4a_5a_6$ such that $d(a_3, P) = 1$ and $d(a_4, Q) = 1$. Say w.l.o.g. that $a_1 \in X_1$, $a_3x_4 \in E$ and $a_4y_1 \in E$. Let σ be a bijection of $V(G)$ such that

$$\begin{aligned}&\sigma(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, a_1, a_2, a_3, a_4, a_5, a_6)\\ &= (a_3, y_2, a_5, x_2, y_3, a_4, x_3, a_2, x_1, y_4, y_1, a_6, a_1, x_4).\end{aligned}$$

It is easy to check that σ is an FPF bipacking of G in a B_{14}, a contradiction.

Therefore $H \cong F_n$ with $n \geq 6$. Let $a_1a_2a_3a_4$ be the path of H with $d_H(a_4) \geq 3$. Let A be the set of endvertices of H that are adjacent to a_4. By Corollary 2.5, no vertex of A is an endendvertex of G. Thus $e(A, P \cup Q) \geq |A| \geq 2$. As $g(G) \geq 8$, we see that $|A| = 2$ and $G[V(P \cup Q) \cup A \cup \{a_4\}]$ is a path of order 11. By Lemma 2.6, $G[V(P \cup Q) \cup A \cup \{a_4\}]$ has an FPF bipacking σ in a B_{11}. Then we readily extend σ to an FPF of G in a B_{15} by setting $\sigma(a_1, a_2, a_3) = (a_3, w, a_1)$, a contradiction. □

Lemma 2.10. *Let $\{x_1y_1, x_2y_2, x_3y_3\}$ be three independent edges in G such that $d(x_i) = 1$ for all $1 \leq i \leq 3$ and either $\{x_1, x_2, x_3\} \subseteq X_1$ or $\{x_1, x_2, x_3\} \subseteq X_2$. Then $G - \{x_1, x_2, x_3\}$ does not have an FPF bipacking in a B_{n-2}.*

Proof. Say $\{x_1, x_2, x_3\} \subseteq X_1$. Let $H = G - \{x_1, x_2, x_3\}$. On the contrary, say H has an FPF bipacking σ in $B_{n-2}(V_1, V_2)$ with $X_1 - \{x_1, x_2, x_3\} \subseteq V_1$ and $X_2 \subseteq V_2$. By Corollary 2.5, $d(y_i) \leq 2$ for all $1 \leq i \leq 3$. Since G does not have an FPF bipacking in a B_{n+1}, it is easy to see that $|X_1| \geq 5$.

We first suppose that $\sigma(\{y_1, y_2, y_3\}) = \{y_1, y_2, y_3\}$. Say w.l.o.g. that

$$\sigma(y_1, y_2, y_3) = (y_2, y_3, y_1).$$

Then we obtain an FPF bipacking of G in $B_{n+1}(V_1 \cup \{x_1, x_2, x_3\}, V_2)$ by extending σ such that $\sigma(x_1) = x_3$, $\sigma(x_2) = x_1$ and $\sigma(x_3) = x_2$ Similarly, if $\{y_i, y_j\} \neq \sigma(\{y_i, y_j\})$ for each $\{i, j\} \subseteq \{1, 2, 3\}$ with $i \neq j$, then we can easily see that σ can be extended to an FPF bipacking of G in a B_{n+1} with $\sigma(\{x_1, x_2, x_3\}) = \{x_1, x_2, x_3\}$. Therefore we may assume w.l.o.g. that $\sigma(y_1) = y_2$, $\sigma(y_2) = y_1$ and $\sigma(y_3) \neq y_3$. Assume for the moment that V_1 has a vertex z such that $y_1 z \notin E(H) \cup E(\sigma(H))$. If $\sigma^{-1}(z)$ does not exist, let $\tau(x_1, x_2, x_3) = (x_3, z, x_1)$ and $\tau(u) = \sigma(u)$ for all $u \in V(H)$. If $\sigma^{-1}(z) = v$ for some $v \in V_1$, let $\tau(v, x_1, x_2, x_3) = (x_1, x_3, z, x_2)$ and $\tau(u) = \sigma(u)$ for all $u \in V(H) - \{v\}$. It is easy to see that τ is an FPF bipacking of G in $B_{n+1}(V_1 \cup \{x_1, x_2, x_3\}, V_2)$ in either case, a contradiction.

Therefore we may assume that $V_1 \subseteq N_H(y_1) \cup N_{\sigma(H)}(y_1)$. As $d_H(y_1) \leq 1$ and $d_{\sigma(H)}(y_1) = d_H(y_2) \leq 1$, we obtain $|V_1| \leq 2$. As $|X_1| \geq 5$, it follows that $|V_1| = 2$, $d_H(y_1) = d_H(y_2) = 1$. Say $V_1 = \{z_1, z_2\}$ with $y_1 z_1 \in E(H)$ and $y_1 z_2 \in E(\sigma(H))$. It follows that $\sigma(z_1) = z_2$, $\sigma(z_2) = z_1$ and $z_1 y_2 \in E(H)$.

If $z_2\sigma(y_3) \notin E(H)$, let $\tau(y_1, y_3, x_1, x_2, x_3) = (\sigma(y_3), y_2, x_2, x_3, x_1)$ and $\tau(u) = \sigma(u)$ for all $u \in V(H) - \{y_1, y_3, x_1, x_2, x_3\}$. Then τ is an FPF bipacking of G in a B_{n+1}, a contradiction. Therefore $z_2\sigma(y_3) \in E(H)$. Then $y_3 z_1 \notin E(H)$. Let w be a new vertex not in G. We may choose an FPF bijection of X_2 such that $\tau(y_1, y_2, y_3) = (y_3, \sigma(y_3), y_1)$, and then extend τ to X_1 such that $\tau(z_1, z_2, x_1, x_2, x_3) = (x_1, w, z_1, x_3, x_2)$. It is easy to see that τ is an FPF bipacking of G in $B_{n+1}(X_1 \cup \{w\}, X_2)$. □

Lemma 2.11. *There exist no three endvertices in G.*

Proof. On the contrary, say that G has three endvertices x_1, x_2 and x_3. We first show that no two of them are adjacent. If this is not the case, say $x_1 x_2 \in E$ with $x_1 \in X_1$. Let $G' = G - \{x_1, x_2\}$. If $G' \in \mathcal{F}$, it is easy to find that G has an FPF bipacking in a B_{n+1}, a contradiction. Therefore $G' \notin \mathcal{F}$ and so G' has an FPF bipacking τ in a $B_{n-1}(V_1, V_2)$ with $X_i - \{x_i\} \subseteq V_i$ for $i \in \{1, 2\}$. We may assume w.l.o.g. that $V_1 = (X_1 - \{x_1\}) \cup \{w\}$ with $w \notin V(G)$. Then $X_2 - \{x_2\} = V_2$. Since $|V_1| = |X_1 - \{x_1\}| + 1$, there exists $v \in V_1$ such that $v \notin \tau(X_1 - \{x_1\})$. If $uv \notin E$ for some $u \in X_2$, then we obtain an FPF bipacking of G in a B_{n+1} by letting $\sigma(x_1, x_2, \tau^{-1}(u)) = (v, u, x_2)$ and $\sigma(z) = \tau(z)$ for all $z \in V(G) - \{x_1, x_2, \tau^{-1}(u)\}$, a contradiction. Therefore $N_G(v) = V_2$. As $g(G) \geq 6$, each vertex of $X_1 - \{v\}$ has degree at most 1. Then by Corollary 2.5, each vertex in $X_2 - \{x_2\}$ has degree at most 2. Then we readily see that G has an FPF bipacking of G in a B_{n+1}, a contradiction.

Therefore no two of x_1, x_2 and x_3 are adjacent. By Corollary 2.4 and Corollary 2.5, there are three vertices y_1, y_2 and y_3 of degree 2, such that $\{x_1 y_1, x_2 y_2, x_3 y_3\} \subseteq E$. We claim that y_1, y_2 and y_3 are distinct. If this is not true, say $y_1 = y_2$. By Lemma 2.1, we see that $G - x_1 - x_2 \in \mathcal{F}$. As y_1 is an isolated vertex of $G - x_1 - x_2$, we see that $G - x_1 - x_2$ consists of two isolated vertices and obviously, G has an FPF bipacking in a B_5, a contradiction. Hence the claim holds.

If $G-\{x_1,x_2,x_3\} \in \mathcal{F}$, then we readily see that either $G-\{x_1,x_2,x_3\} \cong 2K_2$ or $G-\{x_1,x_2,x_3\} \cong F_{n-3}$ by Corollary 2.5 and in this case, we also readily see that G has an FPF bipacking of G in a B_{n+1}, a contradiction. Thus $G-\{x_1,x_2,x_3\} \notin \mathcal{F}$. Then by Lemma 2.10, we obtain $\{x_1,x_2,x_3\} \not\subseteq X_i$ for $i \in \{1,2\}$. Say w.l.o.g. $\{x_1,x_2\} \subseteq X_1$ and $x_3 \in X_2$. Say $N(y_i) = \{x_i, z_i\}$ for $i \in \{1,2,3\}$.

Note that this argument says that neither of X_1 and X_2 contains three endvertices of G.

Let $H = G-\{x_1,x_2,x_3,y_1,y_2,y_3\}$. First, assume that $H \notin \mathcal{F}$. Then H has an FPF bipacking τ in a $B_{n-5}(V_1,V_2)$. If $\{z_1,z_2\} \neq \{\tau(z_1),\tau(z_2)\}$, say $\tau(z_2) \notin \{z_1,z_2\}$, we extend τ to an FPF bipacking of G in a B_{n+1} by letting $\tau(x_1,y_1,x_2,y_2,x_3,y_3) = (x_2,x_3,y_3,y_1,y_2,x_1)$, a contradiction. Therefore $\tau(z_1,z_2) = (z_2,z_1)$. In this situation, we obtain an FPF bipacking of G in a B_{n+1} by letting $\sigma(z_1,x_1,y_1,x_2,y_2,x_3,y_3) = (x_1,y_3,y_2,z_2,x_3,y_1,x_2)$ and $\sigma(u) = \tau(u)$ for all $u \in V(G)-\{z_1,x_1,y_1,x_2,y_2,x_3,y_3\}$, a contradiction.

Therefore $H \in \mathcal{F}$. If $|H| = 2$, it is easy to see that G has an FPF bipacking in a B_9. Assume that $|H| = 4$. Let a_1a_2 and a_3a_4 be the two independent edges of H such that $\{a_1,a_3\} \subseteq X_1$ and if $H \cong P_4$, then $a_2a_3 \in E$. Since X_1 does not contain three endvertices of G, $a_1 \in \{z_1,z_2\}$. Say w.l.o.g. that $a_1 = z_1$. If $a_2a_3 \notin E$, then $z_2 = a_3$ and so G is a linear forest. Consequently, G has an FPF bipacking in a B_{10} by Corollary 2.8, a contradiction. Hence $a_2a_3 \in E$. If $a_4 = z_3$, i.e., $a_4y_3 \in E$, then $x_3y_3a_4a_3$ is feasible and so $x_1y_1a_1a_2$ is not feasible by Lemma 2.9. Thus $z_2 = a_1$. If $z_3 = a_2$ and so a_4 is an endvertex of G and by Corollary 2.5, we see that $z_2 = a_1$. In any case, $G-a_1$ is a linear forest and so $G-a_1$ has an FPF bipacking σ in a B_9, contradicting Corollary 2.8.

Similar to the above argument, it is easy to see that if $H \cong P_6$, then there exists a labelling $H = a_1a_2a_3a_4a_5a_6$ such that $\{y_1a_1, y_2a_1, y_3a_4\} \subseteq E$. Then σ is an FPF bipacking of G in a B_{12} where

$$\begin{aligned}&\sigma(a_1,a_2,a_3,a_4,a_5,a_6,x_1,y_1,x_2,y_2,x_3,y_3)\\ &= (x_1,y_2,a_5,x_3,a_3,y_1,a_1,a_4,y_3,a_2,a_6,x_2),\end{aligned}$$

a contradiction.

Therefore $H \cong F_k$ with $k = n-6 \geq 5$. Since X_i does not contain three endvertices of G for each $i \in \{1,2\}$ and each endvertex of G is adjacent to a vertex of degree 2 in G, it is easy to see that $H \cong P_5$. Furthermore, with Corollary 2.8, we see that there is a labelling $H = a_1a_2a_3a_4a_5$ such that $\{y_1a_2, y_2a_2, y_3a_1\} \subseteq E$. Then σ is an FPF bipacking of G in a B_{12} where

$$\sigma(a_1,a_2,a_3,a_4,a_5,x_1,y_1,x_2,y_2,x_3,y_3) = (y_1,w,a_1,x_2,a_3,a_2,a_5,x_1,x_3,y_2,a_4),$$

a contradiction. □

Corollary 2.12. *The graph G is not a forest.*

Proof. By Corollary 2.4 and Lemma 2.11, we see that if G is a forest, then G is a path. By Lemma 2.2, we conclude that $n \geq 8$. By Lemma 2.6 and Corollary 2.7, there is an FPF bipacking of G in a B_{n+1}, a contradiction. □

Corollary 2.13. *The minimum degree of G is at least* 2.

Proof. On the contrary, let x be an endvertex of G. Say that $x \in X_1$. By Lemma 2.3, there exists $y \in X_2$ such that $d(y) \geq (n_1 - 1)/2$. As G is not a forest and $g(G) \geq 12$, G has a cycle C of order at least 12 and so $n_1 \geq 7$ and $n_2 \geq 6$. Thus $d(y) \geq 3$. Let $Y_1 = N(y)$. By Corollary 2.5, $xy \notin E$ and $d(z) \geq 2$ for each $z \in Y_1$. Clearly, $x \notin V(C)$. If $n_1 = 7$, then $d(y, C) \geq 3$, which implies $G[V(C) \cup \{y\}]$ has a cycle of order less than 12, a contradiction. Hence $n_1 \geq 8$ and so $d(y) \geq 4$. Let $Y_0 = \{y\}$ and $Y_{i+1} = N(Y_i) - Y_{i-1}$ for $i \geq 1$. Let a_1 be the number of endvertices of G contained in Y_2 and a_2 the number of endvertices of G contained in $Y_3 \cup Y_4$. As x is an endvertex of G and by Lemma 2.11, $a_1 \leq 1$ and $a_1 + a_2 \leq 2$. As $g(G) \geq 12$, we see that $|Y_2| \geq |Y_1|$ and $|Y_3| \geq |Y_2| - a_1$ and $|Y_5| \geq |Y_3| - a_2 \geq 2$. Thus $n_1 \geq |Y_1| + |Y_3| + |Y_5| \geq 3|Y_1| - 2a_1 - a_2 \geq 2\lceil (n_1 - 1)/2 \rceil + \lceil (n_1 - 1)/2 \rceil - 2a_1 - a_2$. Since $8 \leq n_1$, $a_1 \leq 1$ and $a_1 + a_2 \leq 2$, we see that $\lceil (n_1 - 1)/2 \rceil \geq 2a_1 + a_2 + 1$ and equality holds only if $8 \leq n_1 \leq 9$ and $a_1 = a_2 = 1$. Clearly, $2\lceil (n_1 - 1)/2 \rceil \geq n_1 - 1$ and equality holds only if n_1 is odd. It follows that $2\lceil (n_1 - 1)/2 \rceil + \lceil (n_1 - 1)/2 \rceil - 2a_1 - a_2 \geq n_1 + \lceil (n_1 - 1)/2 \rceil - 2a_1 - a_2 - 1 \geq n_1$. So equality holds through this equation. This yields that $a_1 = a_2 = 1$, $n_1 = 9$ and every vertex in $Y_1 \cup Y_2 \cup Y_3 \cup Y_4$ has degree 2 if it is not one of the two endvertices. As $|Y_1| \geq 4$, it follows that there are two vertex-disjoint paths of order 4 from Y_1 to Y_4 in $G[Y_1 \cup Y_2 \cup Y_3 \cup Y_4]$, which are two vertex-disjoint feasible paths. This is a contradiction by Lemma 2.9. □

Lemma 2.14. *The minimum degree of G is at least* 3.

Proof. On the contrary, say $\delta(G) = 2$. By Lemma 2.3, for some $\{i, j\} = \{1, 2\}$, there exist two distinct vertices a and b in X_j such that $|N(a) \cup N(b)| \geq (n_i - 1)/2$. We may choose $\{i, j\}$, a and b with $|N(a) \cup N(b)|$ maximal. Subject to this condition, we choose a and b such that the distance $d(a, b)$ from a to b is minimal. Say w.l.o.g. that $\{a, b\} \subseteq X_2$ and $|N(a) \cup N(b)| \geq (n_1 - 1)/2$. Say w.l.o.g. $d(a) \leq d(b)$. As $\delta(G) = 2$ and $g(G) \geq 12$, each component of G contains a cycle of order at least 12. By Corollary 2.7, we see that G has a component which is not a cycle. Thus $\Delta(G) \geq 3$. As $g(G) \geq 8$, we see that $|N(a) \cup N(b)| \geq 5$. Hence $d(b) \geq 3$. We break into the following two cases.

Case 1. $d(a, b) \leq 4$.

Let $c_1 \in N(a)$ and $c_2 \in N(b)$ such that if $d(a, b) = 2$, then $c_1 = c_2$ and if $d(a, b) = 4$, then $N(c_1) \cap N(c_2) \neq \emptyset$. In the latter case, say $N(c_1) \cap N(c_2) = \{c_0\}$. Let $Y_0 = N(b) - \{c_2\}$, $Y_1 = N(Y_0) - \{b\}$ and $Y_{i+1} = N(Y_i) - Y_{i-1}$ for $i = 1, 2, 3$. Since $g(G) \geq 12$ and $\delta(G) \geq 2$, we see that $N(\{a, b, c_1, c_2\}), Y_1, Y_2, Y_3$ and Y_4 are mutually disjoint and $G[N(\{a, b, c_1, c_2\}) \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4]$ is a tree. We use T to denote this tree $G[N(\{a,b,c_1,c_2\}) \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4]$. As $g(G) \geq 12$, we see

$$(1) \qquad |Y_{i+1}| = \sum_{x \in Y_i} (d(x)-1) \quad \text{for } i \in \{0,1,2,3\}.$$

Thus

$$(2) \qquad n_1 \geq |N(\{a,b\})| + |Y_2| + |Y_4| \geq (n_1-1)/2 + |Y_2| + |Y_4|.$$

Consequently,

$$(3) \qquad (n_1+1)/2 \geq |Y_2| + |Y_4|.$$

As $d(b) \geq 3$, $|Y_0| \geq d(b)-1$ and so $|Y_i| \geq d(b)-1 \geq 2$ for $i \in \{1,2,3,4\}$ by (1). Moreover, there are k vertex-disjoint paths $L_1, \dots, L_{k-1}$ and L_k from Y_0 to Y_4, where $k = |Y_0|$. Let u_iv_i with $u_i \in Y_2$ be the second last edge on L_i $(1 \leq i \leq k)$. By Lemma 2.9, at most one of these k edges is a feasible edge. Say w.l.o.g. that u_iv_i is not feasible for $i = 1, \dots, k-1$.

First, assume that u_kv_k is feasible, then by Lemma 2.9, the first edge of L_i is not a feasible edge for each $i \in \{1,2,\dots,k-1\}$. Consequently, $|Y_2| \geq |Y_0| + (k-1)$ and $|Y_4| \geq |Y_0| + 2(k-1)$ by (1). Thus $|Y_2| + |Y_4| \geq 2|Y_0| + 3(k-1)$. If $c_1 = c_2$, then $2|Y_0| \geq |N(a) \cup N(b)| - 1 \geq (n_1-1)/2 - 1$, and so $|Y_2| + |Y_4| > (n_1+1)/2$, contradicting (3). Hence $c_1 \neq c_2$. Then $2|Y_0| \geq d(a) + d(b) - 2 \geq (n_1-1)/2 - 2$ and so

$$(n_1+1)/2 \geq |Y_2| + |Y_4| \geq (n_1-1)/2 - 2 + 3(k-1) \geq (n_1+1)/2.$$

It follows that $k = 2$, i.e., $|Y_0| = 2$ and $d(b) = 3$, $|Y_2| + |Y_4| = (n_1+1)/2$, $|Y_2| = 3$, $|Y_4| = 4$ and $2|Y_0| = d(a) + d(b) - 2$. Consequently, $d(a) = 3$, $|Y_2| + |Y_4| = 3+4 = 7$ and $n_1 = 13$. This means that $X_1 = N(a) \cup N(b) \cup Y_2 \cup Y_4$. Hence $d(c_0) = 2$. As u_kv_k is feasible, c_0c_2 is not feasible by Lemma 2.9. As $X_1 - V(T) = \emptyset$, this implies that there exists $z \in X_2 - \{c_0, b\}$ such that $zc_2 \in E$. As $\delta(G) = 2$ and $g(G) \geq 12$, this implies that $vz \in E$ for some $v \in X_1 - V(T) = \emptyset$, a contradiction.

Therefore u_kv_k is not feasible and so $|Y_4| \geq |Y_2| + k \geq |Y_0| + k$. Thus $|Y_2| + |Y_4| \geq 2|Y_0| + k$. Since $2|Y_0| \geq |N(a) \cup N(b)| - 2 \geq (n_1-1)/2 - 2$ and by (3), it follows that $k \leq 3$ and so $|Y_4| \leq |Y_0| + 3$. As $k \geq 2$, it follows that for some $i \in \{1, \dots, k\}$, the first edge of L_i is feasible for otherwise $|Y_4| \geq |Y_0| + 4$ by (1). If $d(a) = d(b)$, then by symmetry, there exists a feasible edge uv with $u \in N(a) - N(a) \cap N(b)$ and $v \neq a$. As $g(G) \geq 12$, we see that G has two vertex-disjoint feasible paths, a contradiction. Hence $d(a) < d(b)$. Then $2|Y_0| \geq |N(a) \cup N(b)| \geq (n_1-1)/2$ if $c_1 = c_2$ and $2|Y_0| \geq |N(a) \cup N(b)| - 1 \geq (n_1-1)/2 - 1$ if $c_1 \neq c_2$. By (3), it follows that $c_1 \neq c_2$, $d(a) = |Y_0| = k = 2$, $|Y_4| = |Y_0| + 2$ and $X_1 = N(a) \cup N(b) \cup Y_2 \cup Y_4$. Thus the first edge of each L_i is feasible. By Lemma 2.9, c_1c_0 is not feasible. Since $X_1 - V(T) = \emptyset$ and $g(G) \geq 12$, this implies that there exists $z \in X_2 - \{a, c_0\}$ such that $zc_1 \in E$. As

$\delta(G) \geq 2$ and $g(G) \geq 12$, it follows that $vz \in E$ for some $v \in X_1 - V(T) = \emptyset$, a contradiction.

Case 2. $d(a, b) \geq 6$.

Let $Y_0 = N(a), Y_1 = N(Y_0) - \{a\}, Y_2 = N(Y_1) - Y_0$, $Z_0 = N(b), Z_1 = N(Z_0) - \{b\}$, $Z_2 = N(Z_1) - Z_0$ and $J = Y_2 \cap Z_2$. As $d(a, b) \geq 6$, $Y_1 \cap Z_1 = \emptyset$. Let $T_1 = G[\{a\} \cup Y_0 \cup Y_1 \cup Y_2]$, $T_2 = G[\{b\} \cup Z_0 \cup Z_1 \cup Z_2]$. Since $\delta(G) \geq 2$ and $g(G) \geq 12$, $V(T_1) \cap V(T_2) = J$, each of T_1 and T_2 is a tree and each of $E(J, Y_1)$ and $E(J, Z_1)$ consists of $|J|$ independent edges. Furthermore, for each $i \in \{0, 1\}$, $E(Y_i, Y_{i+1})$ contains $|Y_i|$ independent edges, $E(Z_i, Z_{i+1})$ contains $|Z_i|$ independent edges and there are $|J|$ vertex-disjoint paths of order 5 from Y_0 to Z_0 passing through J.

Let E_0 be an edge independent set with $E_0 \subseteq E(Y_0, Y_1)$ and $|E_0| = |Y_0|$. Let F_0 be an edge independent set with $F_0 \subseteq E(Z_0, Z_1)$ and $|F_0| = |Z_0|$. For each edge $xy \in E_0 \cup F_0 \cup E(J, Y_1) \cup E(J, Z_1)$ with $y \in Y_1 \cup Z_1$, we define $\xi(xy)$ to be the subset of $X_1 - Y_0 \cup Z_0$ such that $u \in \xi(xy)$ if and only if $u \in X_1 - Y_0 \cup Z_0$ and either $uy \in E$ with $u \neq x$ or uvx is a path of G for some $v \in X_2 - \{a, b\}$. Since $\delta(G) \geq 2$ and $g(G) \geq 8$, we see that $\xi(e) \neq \emptyset$ for all $e \in E_0 \cup F_0$. Moreover, we see

$$Y_2 = \cup_{e \in E_0} \xi(e) \text{ and } Z_2 = \cup_{e \in F_0} \xi(e); \tag{4}$$

$$|Y_2| = \sum_{e \in E_0} |\xi(e)| \text{ and } |Z_2| = \sum_{e \in F_0} |\xi(e)|. \tag{5}$$

It follows from (4) and (5) that $|Y_2| \geq |Y_0|$ and $|Z_2| \geq |Z_0|$. First, we assume that $Y_2 \cap Z_2 = \emptyset$. Then $n_1 \geq |Y_0| + |Z_0| + |Y_2| + |Z_2| \geq 2(|Y_0| + |Z_0|) = 2|N(\{a, b\})| \geq n_1 - 1$. By (4) and (5), we see that with at most one exception, $|\xi(e)| = 1$, i.e., e is a feasible edge, for all $e \in E_0 \cup F_0$. Thus $E(Y_0, Y_1)$ contains a feasible edge e and $E(Z_0, Z_1)$ contains a feasible edge f and so G has two vertex-disjoint feasible paths, contradicting Lemma 2.9.

Therefore $J \neq \emptyset$. Let $J_0 = \{x \in J \mid d(x) \geq 3\}$, $J_1 = N(J_0) - Y_1 \cup Z_1$ and $J_2 = N(J_1) - J_0$. Since $g(G) \geq 12$, each of $G[V(T_1) \cup (\cup_{i=1}^{2} J_i)]$ and $G[V(T_2) \cup (\cup_{i=1}^{2} J_i)]$ is a tree. Furthermore, we have

$$|J_1| = \sum_{x \in J_0} (d(x) - 2) \text{ and } |J_2| = \sum_{x \in J_1} (d(x) - 1). \tag{6}$$

As $\delta(G) \geq 2$, this implies that $|J_2| \geq |J_1| \geq |J_0|$. If $J_0 = J$, then $n_1 \geq |Y_0| + |Z_0| + |Y_2| + |Z_2| - |J| + |J_2| \geq 2(|Y_0| + |Z_0|) \geq n_1 - 1$. Thus $|\xi(e)| \neq 1$ for at most one edge $e \in E_0 \cup F_0$. That is, with at most one exception, every edge $e \in E_0 \cup F_0$ is a feasible edge of G and consequently, G contains two vertex-disjoint feasible paths, contradicting Lemma 2.9. Therefore $J_0 \neq J$.

Let y be an arbitrary vertex in $Y_1 \cup Z_1$ with $N(y) \cap (J - J_0) \neq \emptyset$. We claim $d(y) \geq 3$. If this is not true, then $d(y) = 2$. Let $u_1u_2u_3u_4u_5$ be a path where $u_1 \in Y_0$, $u_2 \in Y_1$, $u_3 \in J - J_0$, $u_4 \in Z_1$ and $u_5 \in Z_0$ with $y \in \{u_2, u_4\}$. Say w.l.o.g. that $y = u_2$. Then $u_1u_2u_3u_4$ is feasible. By Lemma 2.9, each edge

$e \in E(Y_0 - \{u_1\}, Y_1) \cup E(Z_0 - \{u_5\}), Z_1)$ is not feasible, i.e., $|\xi(e)| \geq 2$. By (4) and (5), we obtain that $|Y_2| \geq 2|Y_0| - 1$ and $|Z_2| \geq 2|Z_0| - 1$. With $|Y_0| \geq |J|$ and $|Z_0| = d(b) \geq 3$, it follows that

$$\begin{aligned} n_1 &\geq |Y_0| + |Z_0| + |Y_2| + |Z_2| - |J| + |J_2| \\ &\geq 2(|Y_0| + |Z_0|) + |Y_0| + |Z_0| - |J| - 2 + |J_2| \\ &\geq (n_1 - 1) + |Y_0| - |J| + |Z_0| - 2 + |J_2| \geq n_1. \end{aligned}$$

This yields that $|Y_0| = |J|$, $d(b) = |Z_0| = 3$, $J_2 = \emptyset$ (i.e., $J_0 = \emptyset$), $|Y_2| = 2|Y_0| - 1$ and $|Z_2| = 2|Z_0| - 1$. Thus $d(u_i) = 2$ for all $i \in \{1, 2, 3, 4, 5\}$. Let $u_6 \in Z_1 - \{u_4\}$ with $u_5u_6 \in E$. Then $u_3u_4u_5u_6$ is a feasible path. Let $u_0 \in Y_0 - \{u_2\}$ with $u_0a \in E$. Then $u_0au_1u_2$ is not a feasible path by Lemma 2.9. Thus $d(a) \neq 2$ and so $d(a) = 3 = d(b)$. Let $v_1 \in J - \{u_3\}$ and $v_2 \in Z_1 - \{u_4\}$ with $v_1v_2 \in E$. By Lemma 2.9, we see that v_1v_2 is not a feasible edge. As $d(v_1) = 2$, this implies $d(v_2) \geq 3$. Clearly, $|N(a) \cup N(b)| \leq |N(a) \cup N(v_2)|$, but $d(a, v_2) = 4 < d(a, b)$, contradicting the minimality of $d(a, b)$. Therefore the claim is true, i.e., $d(y) \geq 3$ for all $y \in Y_1 \cup Z_1$ with $N(y) \cap (J - J_0) \neq \emptyset$. By (4) and (5), this yields that $|Y_2| \geq |Y_0| + |J - J_0|$ and $|Z_2| \geq |Z_0| + |J - J_0|$. Moreover, as $\delta(G) \geq 2$ and $g(G) \geq 12$, there exists a path $x_1x_2x_3x_4x_5$ of order 5 with $x_1 \in J - J_0$, $x_2 \in Y_1$, $x_3 \in Y_2$, $x_3 \notin Y_0 \cup J$, $x_4 \in X_2 - Y_1 \cup Z_1 \cup J_1$ and $x_5 \in X_1$. As $g(G) \geq 12$, we see that $x_5 \notin Y_0 \cup Z_0 \cup Y_2 \cup Z_2 \cup J_2$. Thus

$$n_1 \geq |Y_0| + |Z_0| + |Y_2 \cup Z_2| + |J_2| + 1 \tag{7}$$

$$\geq 2(|Y_0| + |Z_0|) - |J| + 2|J - J_0| + |J_2| + 1 \tag{8}$$

$$\geq n_1 - 1 + |J - J_0| + 1 \geq n_1 + 1, \tag{9}$$

a contradiction. □

We are now ready to complete the proof of the theorem. Choose $x \in X_1$ such that $d(x) = \Delta_1$. Let $A_0 = \{x\}$ and $A_1 = N(x)$. For each $i \in \{2, 3, 4, 5\}$, let $A_i = N(A_{i-1}) - A_{i-2}$. Since $g(G) \geq 12$, $A_i \cap A_j = \emptyset$ for all $0 \leq i < j \leq 5$ and $|A_i| = \sum_{y \in A_{i-1}} (d(y) - 1)$ for each $i \in \{2, 3, 4, 5\}$. Thus if $A_i \subseteq X_1$, then $|A_i| \geq |A_{i-1}|(\delta_2 - 1)$ and if $A_i \subseteq X_2$, then $|A_i| \geq |A_{i-1}|(\delta_1 - 1)$ for each $i \in \{2, 3, 4, 5\}$. As $A_5 \subseteq X_2$, we obtain $n_2 \geq |A_5| \geq |A_1|(\delta_2 - 1)^2(\delta_1 - 1)^2 = \Delta_1(\delta_2 - 1)^2(\delta_1 - 1)^2$. Since $\delta \geq 3$, $(\delta_2 - 1)^2 \geq \delta_2 + 1$ and $(\delta_1 - 1)^2 \geq 4$. Consequently, $n_2 \geq 4(\delta_2 + 1)\Delta_1 > 1 + 2\delta_2\Delta_1$, contradicting Corollary 2.4. This proves the theorem.

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