This article is published in Graphs and Combinatorics. The original publication is available at http://www.springerlink.com.

Proof of the Erdős-Faudree Conjecture on Quadrilaterals

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In this paper, we prove Erdős-Faudree's conjecture: If G is a graph of order $4k$ and the minimum degree of G is at least $2k$ then G contains k disjoint cycles of length 4.

Key words: 4-cycles, disjoint cycles, cycle coverings

1 Introduction and Notation

Let G be a graph. A set of graphs are said to be disjoint if no two of them have any common vertex. Corrádi and Hajnal [2] investigated the maximum number of disjoint cycles in a graph. They proved that if G is a graph of order at least $3k$ with minimum degree at least $2k$, then G contains k disjoint cycles. In particular, when the order of G is exactly $3k$, then G contains k disjoint triangles. Erdős and Faudree [4] conjectured that if G is a graph of order $4k$ with minimum degree at least $2k$, then G contains k disjoint cycles of length 4. With respect to this conjecture, Randerath, Schiermeyer and Wang [6] proved that G contains $k-1$ cycles of length 4 and a subgraph of order 4 with at least four edges such that all of them are disjoint. In [7], we improved this result by showing the following result:

Theorem *A Let G be a graph of order n with $4k+1 \le n \le 4k+4$, where k is a positive integer. Suppose that the minimum degree of G is at least $2k+1$. Then G contains at least k disjoint cycles of length 4.*

El-Zahar [3] conjectured that if G is a graph of order $n = n_1+n_2+\cdots+n_k$ with $n_i \ge 3 (1 \le i \le k)$ and the minimum degree of G is at least $\lceil n_1/2\rceil + \lceil n_2/2\rceil + \cdots + \lceil n_k/2\rceil$, then G contains k disjoint cycles of lengths $n_1, n_2, \ldots, n_k$, respectively. He proved this conjecture for $k = 2$. When $n_1 = n_2 = \cdots = n_k = 4$, El-Zahar's conjecture reduces to the above conjecture of Erdős and Faudree. Komlós, Sárközy and Szemerédi [5] showed that for any graph H of order r with chromatic number k, there exist constants

c and n_0 such that if $n \geq n_0$, $r|n$ and G is a graph of order n with minimum degree at least $(1-1/k)n+c$ then G contains n/r disjoint copies of H. In this paper, we prove the following theorem:

Theorem *B If G is a graph of order $4k$ and the minimum degree of G is at least $2k$ then G contains k disjoint cycles of length 4.*

We shall use the terminology and notation from [1] except as indicated. Let G be a graph. Let $u \in V(G)$. The neighborhood of u in G is denoted by $N(u)$. Let H be a subgraph of G or a subset of $V(G)$ or a sequence of distinct vertices of G. We define $N(u,H)$ to be the set of neighbors of u contained in H, and let $e(u,H) = |N(u,H)|$. Clearly, $N(u,G) = N(u)$ and $e(u,G)$ is the degree of u in G. If X is a subgraph of G or a subset of $V(G)$ or a sequence of distinct vertices of G, we define $N(X,H) = \cup_u N(u,H)$ and $e(X,H) = \sum_u e(u,H)$ where u runs over all the vertices in X. Let x and y be two distinct vertices. We define $I(xy,H)$ to be $N(x,H) \cap N(y,H)$ and let $i(xy,H) = |I(xy,H)|$. Let each of $X_1, X_2, \ldots, X_r$ be a subgraph of G or a subset of $V(G)$. We use $[X_1, X_2, \ldots, X_r]$ to denote the subgraph of G induced by the set of all the vertices that belong to at least one of $X_1, X_2, \ldots, X_r$. We use C_i to denote a cycle of length i for all integers $i \geq 3$, and use P_j to denote a path of order j for all integers $j \geq 1$. For a cycle C of G, a chord of C is an edge of $G - E(C)$ which joins two vertices of C, and we use $\tau(C)$ to denote the number of chords of C in G. An n-cycle is a cycle of length n. Clearly, if C is a 4-cycle then $\tau(C) \in \{0,1,2\}$.

We use C_4^+ to denote a graph of order 4 with five edges. Obviously, C_4^+ can be obtained from K_4 by deleting one edge from K_4. If F is a graph of order 4 and size 4 with a triangle, we may write F as a trail $x_0x_1x_2x_3x_1$.

If S is a set of subgraphs of G, we write $G \supseteq S$. For an integer $k \geq 1$ and a graph G', we use kG' to denote a set of k disjoint graphs isomorphic to G'. If $G_1, \ldots, G_r$ are r graphs and $k_1, \ldots, k_r$ are r positive integers, we use $k_1G_1 \uplus \cdots \uplus k_rG_r$ to denote a set of $k_1 + \cdots + k_r$ disjoint graphs which consist of k_1 copies of $G_1, \ldots, k_{r-1}$ copies of G_{r-1} and k_r copies of G_r. For two graphs H_1 and H_2, the union of H_1 and H_2 is still denoted by $H_1 \cup H_2$ as usual, that is, $H_1 \cup H_2 = (V(H_1) \cup V(H_2), E(H_1) \cup E(H_2))$. Let each of Y and Z be a subgraph of G, or a subset of $V(G)$, or a sequence of distinct vertices of G. If Y and Z do not have any common vertices, we define $E(Y,Z)$ to be the set of all the edges of G between Y and Z. Clearly, $e(Y,Z) = |E(Y,Z)|$. If $C = x_1x_2 \ldots x_rx_1$ is a cycle, then the operations on the subscripts of the x_i's will be taken by modulo r in $\{1,2,\ldots,r\}$. If C is a 4-cycle and $u \in V(C)$, we use u^* to denote the unique vertex of C such that u and u^* are not consecutive on C. For two graphs G and H, we write $G \cap H = \emptyset$ if G and H are disjoint.

Let $\{H, Q_1, \ldots, Q_t\}$ be a set of $t+1$ disjoint subgraphs of G such that $Q_i \cong C_4$

for $i = 1, \ldots, t$. We say that $\{H, Q_1, \ldots, Q_t\}$ is optimal if $[H, Q_1, \ldots, Q_t]$ does not contain $t+1$ disjoint subgraphs $H', Q'_1, \ldots, Q'_t$ such that $H' \cong H$, $Q'_i \cong C_4 (1 \leq i \leq t)$ and $\sum_{i=1}^t \tau(Q'_i) > \sum_{i=1}^t \tau(Q_i)$. Let Q be a 4-cycle and H a subgraph of order 4 in G. We write $H \geq Q$ if H has a 4-cycle Q' such that $\tau(Q') \geq \tau(Q)$. Moreover, if $\tau(Q') > \tau(Q)$, we write $H > Q$.

Let Q be a 4-cycle of G and $u \in V(Q)$. Let $x \in V(G) - V(Q)$. We write $x \to (Q, u)$ if $[Q - u + x] \supseteq C_4$. In this case, we say that u is replaceable by x in Q. Moreover, if $[Q - u + x] \geq Q$ then we write $x \Rightarrow (Q, u)$ and if $[Q - u + x] > Q$ then we write $x \stackrel{a}{\to} (Q, u)$. In addition, if it does not hold that $x \stackrel{a}{\to} (Q, u)$ then we write $x \stackrel{na}{\to} (Q, u)$. Clearly, $x \Rightarrow (Q, u)$ when $x \stackrel{a}{\to} (Q, u)$. If $x \to (Q, u)$ for all $u \in V(Q)$ then we write $x \to Q$. Similarly, we define $x \Rightarrow Q$. Note that if $e(x, Q) = 3$ then $x \to Q$ if and only if $dd^* \in E$ where $d \in V(Q)$ with $xd \notin E$.

Let P be a path of order at least 2 or a sequence of at least two distinct vertices in $G - V(Q + x)$. Let X be a subset of $V(G) - V(Q + x)$ with $|X| \geq 2$. We write $x \to (Q, u; P)$ if $x \to (Q, u)$ and u is adjacent to the two end vertices of P. In this case, if P is a path of order 3, then $[x, Q, P] \supseteq 2C_4$. We write $x \to (Q, u; X)$ if $x \to (Q, u; yz)$ for some $\{y, z\} \subseteq X$ with $y \neq z$. We write $x \to (Q; P)$ if $x \to (Q, u; P)$ for some $u \in V(Q)$. Similarly, we define $x \to (Q; X)$.

We use "w.l.o.g." for "without loss of generality" and "w.r.t." for "with respect to".

2 Sketch of the Proof of Theorem B

Let $G = (V, E)$ be a graph of order $4k$ with minimum degree at least $2k$. Suppose, for a contradiction, that $G \not\supseteq kC_4$. By the result of [6] mentioned in the introduction, there exists a sequence $(T, Q_1, \ldots, Q_{k-1})$ of k disjoint subgraphs such that $T \cong C_3$ and $Q_i \cong C_4$ for $i = 1, \ldots, k-1$. We call such a sequence $(T, Q_1, \ldots, Q_{k-1})$ a chain of G. Among all the chains of G, we choose $(T, Q_1, \ldots, Q_{k-1})$ such that

$$\sum_{i=1}^{k-1} \tau(Q_i) \text{ is maximum.} \tag{1}$$

Subject to (1), we further choose $(T, Q_1, \ldots, Q_{k-1})$ such that

$$|\{Q_i | \tau(Q_i) = 2, 1 \leq i \leq k-1\}| \text{ is maximum.} \tag{2}$$

A chain satisfying (1) and (2) is called a *feasible chain* of G. If $(T, Q_1, \ldots, Q_{k-1})$ is a feasible chain, we define the terminal point of $(T, Q_1, \ldots, Q_{k-1})$ to be the unique vertex of G which does not belong to $V(T) \cup V(\cup_{i=1}^{k-1} Q_i)$. A *strong feasible chain* of G

is a sequence $(xy, T, Q_1, \ldots, Q_{k-1})$ of subgraphs of G such that $(T, Q_1, \ldots, Q_{k-1})$ is a feasible chain of G, $xy \in E$, $y \in V(T)$ and x is the terminal point of $(T, Q_1, \ldots, Q_{k-1})$. The following Claims 2.1-2.7 will be proved in Section 4. Claims 2.1-2.4 are steps towards Claims 2.5-2.7. We derive Theorem B from Claims 2.5-2.7 in this section. Our first important step is the following Claim 2.1.

Claim 2.1. *There exists a strong feasible chain in G.*

By Claim 2.1, let $\sigma = (x_0x_1, T, Q_1, \ldots, Q_{k-1})$ be any given strong feasible chain with $x_1 \in V(T)$. Let $T = x_1x_2x_3x_1$, $F = x_0x_1x_2x_3x_1$ and $\mathcal{Q} = \{Q_1, \ldots, Q_{k-1}\}$.

Claim 2.2. *For each $Q \in \mathcal{Q}$, if $e(F, Q) \geq 9$ then either $e(x_0, Q) = 0$ or there exists a labelling $Q = a_1a_2a_3a_4a_1$ such that $N(x_0, Q) = \{a_1\}$, $e(x_1, Q) = 4$, $N(x_2, Q) = \{a_1, a_4\}$, $N(x_3, Q) = \{a_1, a_2\}$, $a_1a_3 \in E$ and $a_2a_4 \notin E$.*

Claim 2.3. *For each $Q \in \mathcal{Q}$, if $e(x_0, Q) = 4$ and $e(x_1, Q) \geq 1$ then $e(x_2, Q) \leq 1$ and $e(x_3, Q) \leq 1$.*

Claim 2.4. *For each $Q \in \mathcal{Q}$, $e(x_0x_2, Q) \leq 6$ and $e(x_0x_3, Q) \leq 6$.*

Claim 2.5. *For each $Q \in \mathcal{Q}$, if $e(F - x_1, Q) \geq 7$ then either $e(x_0, Q) = 0$ or $e(x_0, Q) = 1$, $e(x_2x_3, Q) = 6$, $N(x_2, Q) = N(x_3, Q)$.*

Claim 2.6. *For each $Q \in \mathcal{Q}$, if $e(x_0, Q) = 4$ then $e(x_2x_3, Q) = 0$.*

Claim 2.7. *For each $Q \in \mathcal{Q}$, if $e(x_0, Q) = 3$ then $e(x_2x_3, Q) \leq 2$.*

Proof of Theorem B**.** Clearly, $e(x_0, G - V(F)) + e(F - x_1, G - V(F)) \geq 8k - 6 = 8(k-1) + 2$. Thus $e(x_0, Q) + e(F - x_1, Q) \geq 9$ for some $Q \in \mathcal{Q}$. If $e(x_0, Q) = 4$ then $e(x_2x_3, Q) = 0$ by Claim 2.6 and so $e(x_0, Q) + e(F - x_1, Q) = 8$, a contradiction. If $e(x_0, Q) = 3$ then $e(x_2x_3, Q) \leq 2$ by Claim 2.7 and so $e(x_0, Q) + e(F - x_1, Q) \leq 8$, a contradiction. Hence $e(x_0, Q) \leq 2$. Thus $e(F - x_1, Q) \geq 7$. By Claim 2.5, either $e(x_0, Q) = 0$ or $e(x_0, Q) = 1$ with $e(x_2x_3, Q) = 6$. Then $e(x_0, Q) + e(F - x_1, Q) \leq 8$, a contradiction. ∎

3 Preliminary Lemmas

Let $G = (V, E)$ be a given graph in the following. Lemma 3.1 is an easy observation.

Lemma 3.1 *Let T and Q be two disjoint subgraphs of G with $T \cong C_3$ and $Q \cong K_4$ such that $e(T, Q) \geq 11$. Let x_1 and x_2 be two distinct vertices of T. Set $G_0 = [T, Q]$. Then the following statements hold:*

(a) For each $x \in V(G) - V(G_0)$ with $e(x, G_0) \geq 2$, $[G_0, x] \supseteq 2C_4$

(b) For each edge $uv \in E(G_0 - \{x_1, x_2\})$, there exists a triangle T' in $G_0 - \{x_1, x_2\}$ such that $uv \in E(T')$ and $G_0 - V(T') \cong K_4$.

(c) *There exists a labelling* $V(Q) = \{b_1, b_2, b_3, b_4\}$ *such that* $\{b_1, b_2, b_3\} \subseteq N(x_1, Q)$ *and* $\{x_2, x_3, b_4, b_r\} \cong K_4$ *for* $r = 2, 3$.

(d) *Let* $Z \subseteq V(G_0 - \{x_1, x_2\})$ *with* $|Z| = 4$. *If* $e(x_1x_2, Q) = 8$ *then there exists a triangle* T' *in* $[Z]$ *such that* $G_0 - V(T') \cong K_4$.

Lemma 3.2 *Let* T *and* Q *be two disjoint subgraphs of* G *and* $z \in V(G) - V(T \cup Q)$ *such that* $T \cong C_3$, $Q \cong C_4$, $e(T, Q) \geq 9$ *and* $[T, Q, z] \not\supseteq 2C_4$. *Suppose that* $[T, Q, z] \not\supseteq C$ *with* $C \cong C_3$ *and* $[T, Q, z] - V(C) > Q$. *Then* $e(z, Q) \leq 1$.

Proof. Say $Q = d_1d_2d_3d_4d_1$. Suppose $e(z, Q) \geq 2$. As $[T, Q, z] \not\supseteq 2C$, $z \not\to (Q; V(T))$. As $e(T, Q) \geq 9$, for each $i \in \{1, 2\}$, $e(d_r, T) \geq 2$ for some $r \in \{i-1, i+1\}$ and so $e(z, d_id_{i+2}) \leq 1$. Thus we may assume $e(z, d_1d_2) = 2$. Then $[z, d_1, d_2] \supseteq C_3$. As $e(d_3d_4, T) \geq 3$, $[T, d_3, d_4] \supseteq C_4^+$. Thus $\tau(Q) \geq 1$. Say w.l.o.g. $d_1d_3 \in E$. Then $e(d_4, T) \leq 1$ and $d_2d_4 \notin E$ as $z \not\to (Q; V(T))$. It follows that $e(d_3, T) = 3$ or $e(d_2, T) = 3$. Then $[T, Q, z] \supseteq C_3 \uplus K_4$ and so $\tau(Q) = 2$, a contradiction. ∎

Lemma 3.3 *Let* $F = x_0x_1x_2x_3x_1$, Q *a* 4*-cycle of* $G - V(F)$ *and* $z \in V(G) - V(F \cup Q)$ *such that* $z \not\to (Q; x_2x_3)$. *Suppose that* $[F, Q] \not\supseteq P \uplus Q'$ *with* $P \supseteq 2P_2$ *and* $\tau(Q') = \tau(Q) + 2$. *Furthermore, suppose that* $e(x_0x_2x_3z, Q) \geq 9$ *such that either* $e(x_0, Q) = 1$ *and* $e(x_2x_3, Q) = 6$ *with* $N(x_2, Q) = N(x_3, Q)$ *or* $e(x_0, Q) = 0$ *with* $e(x_2x_3, Q) \geq 7$. *Then* $e(x_0x_2x_3z, Q) = 9$ *and there exists a labelling* $Q = d_1d_2d_3d_4d_1$ *such that* $e(x_2x_3, d_2d_3d_4) = 6$ *and* $zd_3 \in E$.

Proof. Say $Q = d_1d_2d_3d_4d_1$. If $e(x_0, Q) = 1$, we may assume that $N(x_2, Q) = N(x_3, Q) = \{d_2, d_3, d_4\}$. It is easy to see that $[x_2, x_3, Q] \supseteq P_2 \uplus K_4$ regardless $e(x_0, Q) = 0$ or $e(x_0, Q) = 1$. Thus $[F, Q] \supseteq 2P_2 \uplus K_4$. Then $\tau(Q) \neq 0$ by our assumption. As $z \not\to (Q; x_2x_3)$, it follows that if $e(x_0, Q) = 1$ then $d_2d_4 \in E$, $d_1d_3 \notin E$ and $N(z, Q) = \{d_3, d_i\}$ for some $i \in \{2, 4\}$. Thus the lemma holds. So assume $e(x_0, Q) = 0$. If $e(z, Q) = 1$ then $e(x_2x_3, Q) = 8$ and obviously the lemma holds. So assume $e(z, Q) \geq 2$. For each $i \in \{1, 2\}$, $e(d_r, x_2x_3) = 2$ for some $r \in \{i-1, i+1\}$ and so $e(z, d_id_{i+2}) \leq 1$ since $z \not\to (Q; x_2x_3)$. Therefore $N(z, Q) = \{d_i, d_{i+1}\}$ for some $i \in \{1, 2, 3, 4\}$. Say w.l.o.g. $N(z, Q) = \{d_3, d_4\}$. As $\tau(Q) \geq 1$, say w.l.o.g. $d_2d_4 \in E$. Then $e(d_1, x_2x_3) \neq 2$ as $z \not\to (Q, d_1; x_2x_3)$. Thus $e(d_1, x_2x_3) = 1$, $e(x_2x_3, d_2d_3d_4) = 6$ and so the lemma holds. ∎

Lemma 3.4 *Let* $F = x_0x_1x_2x_3x_1$ *and* Q *be two disjoint subgraphs with* $Q \cong C_4$. *The following two statements hold:*

(a) (*Lemma* 2.7, [7]) *If* $e(F, Q) \geq 11$ *and* $e(x_0, Q) \geq 1$ *then* $[F, Q] \supseteq 2C_4$, *or there exists a labelling* $Q = a_1a_2a_3a_4a_1$ *such that* $N(x_0, Q) = \{a_1, a_2, a_3\}$, $e(x_1, Q) = 4$ *and* $N(x_2, Q) = N(x_3, Q) = \{a_1, a_3\}$.

(b) *If* $e(x_0, Q) \geq 1$, $e(x_1x_2x_3, Q) \geq 9$, $\tau(Q) \geq 1$ *and* $x_i \to Q$ *for some* $i \in \{1, 2, 3\}$ *then* $[F, Q] \supseteq 2C_4$.

Proof. We only need to show (b) here. Suppose that $[F,Q] \not\supseteq 2C_4$. Say $Q = d_1d_2d_3d_4d_1$ with $d_1d_3 \in E$. First, assume that $e(x_0, d_2d_4) \geq 1$. W.l.o.g., say $x_0d_4 \in E$. As $e(x_2x_3, Q) \geq 9 - e(x_1, Q) \geq 5$, $e(x_i, d_1d_2d_3) \geq 2$ for some $i \in \{2,3\}$. Say w.l.o.g. $e(x_2, d_1d_2d_3) \geq 2$. As $[F,Q] \not\supseteq 2C_4$, $x_2 \not\to (Q, d_4; x_0x_1x_3)$ and so $d_4x_3 \notin E$. If we also have $e(x_3, d_1d_2d_3) \geq 2$ then $d_4x_2 \notin E$. Consequently, $6 \geq e(x_2x_3, Q) \geq 5$ and so $e(x_1, Q) \geq 3$. Thus $e(x_1, d_1d_3) \geq 1$. Say w.l.o.g. $x_1d_1 \in E$. Then $[x_0, d_4, d_1, x_1] \supseteq C_4$ and $[x_2, x_3, d_2, d_3] \supseteq C_4$, a contradiction. Hence $e(x_3, d_1d_2d_3) \leq 1$. It follows that $e(x_1x_2, Q) = 8$ and $e(x_3, Q) = 1$. Then $[x_0, d_4, d_r, x_1] \supseteq C_4$ and $[x_2, x_3, d_2, d_t] \supseteq C_4$ where $\{r,t\} = \{1,3\}$ and $e(x_3, d_2d_t) = 1$, a contradiction. Therefore $e(x_0, d_1d_3) \geq 1$. Similarly, if $d_2d_4 \in E$ then $[F,Q] \supseteq 2C_4$, a contradiction. Hence $d_2d_4 \notin E$. Say w.l.o.g. $x_0d_1 \in E$. Suppose that $e(x_i, d_2d_4) = 2$ for some $i \in \{2,3\}$. W.l.o.g., say $e(x_2, d_2d_4) = 2$. Then $x_3d_1 \notin E$ as $x_2 \not\to (Q, d_1; x_0x_1x_3)$. If $e(x_1, d_2d_4) = 0$ then $e(x_1, d_1d_3) = 2$, $e(x_2, Q) = 4$ and $e(x_3, d_2d_3d_4) = 3$. Consequently, $[x_0, d_1, d_3, x_1] \supseteq C_4$ and $[x_2, d_2, x_3, d_4] \supseteq C_4$, a contradiction. Hence $e(x_1, d_2d_4) \geq 1$. Say w.l.o.g. $x_1d_4 \in E$. Then $[x_0, d_1, d_4, x_1] \supseteq C_4$ and so $[x_2, x_3, d_2, d_3] \not\supseteq C_4$. This implies that $e(x_2x_3, d_2d_3) \leq 2$. As $e(x_1x_2x_3, Q) \geq 9$, it follows that $e(x_1, Q) = 4$, $x_3d_4 \in E$ and $e(x_2x_3, d_2d_3) = 2$. Then $[x_0, d_1, d_2, x_1] \supseteq C_4$ and so $[x_2, x_3, d_3, d_4] \not\supseteq C_4$. This yields $e(d_3, x_2x_3) = 0$. It follows that $e(x_3, d_2d_4) = 2$ as $e(x_2x_3, Q) \geq 5$. Thus $[x_0, d_1, d_3, x_1] \supseteq C_4$ and $[x_2, d_2, x_3, d_4] \supseteq C_4$, a contradiction. Therefore $e(x_i, d_2d_4) \leq 1$ for each $i \in \{2,3\}$. Hence $x_i \not\to Q$ for $i \in \{2,3\}$. Thus $x_1 \to Q$. This implies that $\{d_2, d_4\} \subseteq N(x_1)$. As $e(x_2x_3, Q) \geq 5$, say w.l.o.g. $e(x_2, d_1d_2d_3) = 3$. As $[x_0, d_1, d_i, x_1] \supseteq C_4$ for each $i \in \{2,4\}$, $[x_2, x_3, d_3, d_i] \not\supseteq C_4$ for each $i \in \{2,4\}$. This implies that $e(x_3, d_2d_3d_4) = 0$ and so $e(x_2x_3, Q) \leq 4$, a contradiction. ∎

Lemma 3.5 *Let P be a path of order 4 and Q a 4-cycle of G such that $P \cap Q = \emptyset$ and $\{P, Q\}$ is optimal. If $e(P,Q) \geq 9$ and $[P,Q] \not\supseteq 2C_4$ then either $[P,Q]$ contains two disjoint subgraphs T and C such that $T \cong C_3$, $C \cong C_4$ and $\tau(C) \geq \tau(Q)$, or $\tau(Q) = 2$ and there exist two labellings $P = y_1y_2y_3y_4$ and $V(Q) = \{b_1, b_2, b_3, b_4\}$ such that one of the following two statements (a) and (b) holds:*

(a) $N(y_1, Q) \cup N(y_3, Q) \subseteq \{b_1, b_2, b_3\}$, $3 \leq e(y_2, Q) \leq 4$, $e(y_4, Q) = 0$, $e(P,Q) \leq 10$;

(b) $N(y_1, Q) \cup N(y_4, Q) \subseteq \{b_1, b_2\}$, $N(y_2, Q) \cup N(y_3, Q) \subseteq \{b_1, b_2, b_3\}$, $e(P,Q) \leq 10$.

In addition, if (a) holds, then $y_i \to (Q; y_jy_l)$ for each $\{i,j,l\} = \{1,2,3\}$. If (b) holds, then $e(y_i, Q) = 3$ for some $i \in \{2,3\}$ and $y_i \to (Q; y_jy_l)$ for each $\{j,l\} \subseteq \{1,2,3,4\} - \{i\}$ with $j \neq l$. Furthermore, if any of (a) and (b) holds then $[P,Q] \supseteq C_3 \uplus C_4^+$.

Proof. Let $P = y_1y_2y_3y_4$, $Q = b_1b_2b_3b_4b_1$ and $H = [P,Q]$. For the proof, suppose that H does not contain the two described subgraphs T and C. We shall prove that one of (a) and (b) holds. We divide the proof into the two cases: $\tau(Q) \leq 1$ or $\tau(Q) = 2$.

Case 1. $\tau(Q) \leq 1$.

In this case, $H \not\supseteq C_3 \uplus C_4^+$ and $H \not\supseteq P_4 \uplus K_4$ by the assumption of the lemma. As $e(y_1y_2, Q) + e(y_3y_4, Q) \geq 9$, we may assume w.l.o.g. that $e(y_1y_2, Q) \geq 5$. Then $e(y_1y_2, b_1b_2) \geq 3$ or $e(y_1y_2, b_3b_4) \geq 3$. W.l.o.g., say the former holds. Then $[y_1, y_2, b_1, b_2] \supseteq C_4^+$. As $H \not\supseteq C_3 \uplus C_4^+$ and $H \not\supseteq 2C_4$, we see $e(y_3y_4, b_3b_4) \leq 1$. If we also have $e(y_1y_2, b_3b_4) \geq 3$, then $e(y_3y_4, b_1b_2) \leq 1$ and so $e(y_1y_2, Q) \geq 7$. Thus either $[y_1, y_2, b_1, b_2] \cong K_4$ or $[y_1, y_2, b_3, b_4] \cong K_4$. W.l.o.g., say the former holds. Then $e(y_3y_4, b_3b_4) = 0$ as $H \not\supseteq P_4 \uplus K_4$. Thus $e(y_1y_2, Q) = 8$, $e(y_3y_4, b_1b_2) = 1$ and so $H \supseteq P_4 \uplus K_4$, a contradiction. Hence $e(y_1y_2, b_3b_4) \leq 2$. If $e(y_3y_4, b_1b_2) \geq 3$, then we also have that $e(y_1y_2, b_3b_4) \leq 1$ and so $e(b_1b_2, P) \geq 7$. Consequently, either $[b_1, b_2, y_1, y_2] \cong K_4$ or $[b_1, b_2, y_3, y_4] \cong K_4$. W.l.o.g., say the former holds. Then $e(y_3y_4, b_3b_4) = 0$ as $H \not\supseteq P_4 \uplus K_4$. Thus $e(b_1b_2, P) = 8$, $e(y_1y_2, b_3b_4) = 1$ and so $H \supseteq P_4 \uplus K_4$, a contradiction. We conclude that $e(y_3y_4, b_1b_2) \leq 2$. As $e(P, Q) \geq 9$, it follows that $e(y_1y_2, b_1b_2) = 4$ and $e(y_3y_4, b_3b_4) = 1$. Thus $H \supseteq P_4 \uplus K_4$, a contradiction.

Case 2. $\tau(Q) = 2$.

W.l.o.g., say $e(y_1, Q) \geq e(y_4, Q)$. Then $e(y_1, Q) \geq 1$. Suppose that $e(y_1, Q) = 4$. As $H \not\supseteq 2C_4$ and $H \not\supseteq C_3 \uplus K_4$, $e(b_i, P - y_1) \leq 1$ for each $b_i \in V(Q)$. Thus $e(P, Q) \leq 8$, a contradiction. Hence $e(y_1, Q) \leq 3$.

Suppose $e(y_1y_4, Q) \leq 2$. Then $e(y_2y_3, Q) \geq 7$. If $e(y_4, Q) = 1$, then $e(y_1, Q) = 1$ and it is easy to see that if $N(y_1, Q) \neq N(y_4, Q)$ then $H \supseteq 2C_4$. Moreover, if $N(y_1, Q) = N(y_4, Q)$, say w.l.o.g. $e(b_1, y_1y_4) = 2$, then $y_ib_1 \in E$ and $e(y_j, b_2b_3b_4) = 3$ for some $\{i, j\} = \{2, 3\}$. Consequently, $H \supseteq C_3 \uplus K_4$, a contradiction. Hence $e(y_4, Q) = 0$. If $e(y_1, Q) = 1$ then $e(y_2y_3, Q) = 8$ and so $H \supseteq C_3 \uplus K_4$, a contradiction. Hence we may assume that $e(y_1, b_1b_2) = 2$. If $\{b_3, b_4\} \subseteq N(y_3)$ then it is easy to see that for some $i \in \{1, 2\}$, $e(y_3, Q - b_i) = 3$, $y_2b_i \in E$ and so $H \supseteq C_3 \uplus K_4$, a contradiction. Hence $\{b_3, b_4\} \not\subseteq N(y_3)$. Say w.l.o.g. $y_3b_4 \notin E$. Then $e(y_2, Q) = 4$, $e(y_3, b_1b_2b_3) = 3$ and so (a) holds. Therefore we may assume $e(y_1y_4, Q) \geq 3$ and so $e(y_1, Q) \geq 2$ in the following.

Suppose $e(y_1, Q) = 2$. Say w.l.o.g. $e(y_1, b_1b_2) = 2$. We claim $e(y_4, b_3b_4) = 0$. If this is false, say w.l.o.g. $y_4b_4 \in E$. Then $y_2b_4 \notin E$ as $H \not\supseteq 2C_4$. Then $e(y_2y_3, b_1b_2b_3) \geq 9 - e(y_1y_4, Q) - e(y_3, b_4) \geq 9 - 4 - 1 = 4$. If $e(b_3, y_2y_3) = 0$ then $e(y_2y_3, b_1b_2) = 4$ and $y_3b_4 \in E$. Consequently, $[y_1, y_2, b_1, b_2] \supseteq K_4$ and $[y_3, y_4, b_4] \supseteq C_3$, a contradiction. Hence $e(b_3, y_2y_3) \geq 1$. It follows that either $E(y_2y_3, b_1b_3)$ or $E(y_2y_3, b_2b_3)$ contains two independent edges. Then we readily see that $H \supseteq 2C_4$, a contradiction. Hence $e(y_4, b_3b_4) = 0$ and so $e(y_4, b_1b_2) \geq 1$. If $N(y_2y_3, Q) \subseteq \{b_1, b_2, b_i\}$ for some $i \in \{3, 4\}$, then we may assume w.l.o.g. $i = 3$ and so (b) holds. Therefore we may assume that $e(b_i, y_2y_3) \geq 1$ for $i \in \{3, 4\}$. Since $E(y_1y_4, b_1b_2)$ contains two independent edges, $E(y_2y_3, b_3b_4)$ does not contain two independent edges for otherwise $H \supseteq 2C_4$. Thus

$E(y_2y_3, b_3b_4) = \{y_rb_3, y_rb_4\}$ for some $r \in \{2,3\}$. Then $e(y_4y_2y_3, b_1b_2) \geq 5$. Thus $y_2b_i \in E$ and $e(b_j, y_3y_4) = 2$ for some $\{i,j\} = \{1,2\}$. Then $[y_1, y_2, b_i] \supseteq C_3$ and $[y_3, y_4, b_j] \supseteq C_3$. As $H \not\supseteq C_3 \uplus K_4$, this implies that $[y_3, b_j, b_3, b_4] \not\supseteq K_4$ and $[y_2, b_3, b_4, b_i] \not\supseteq K_4$. This yields that $e(y_3, b_3b_4) \leq 1$ and $e(y_2, b_3b_4) \leq 1$, a contradiction. Finally, suppose $e(y_1, Q) = 3$. Say $e(y_1, b_1b_2b_3) = 3$. Then $e(b_4, y_2y_3y_4) \leq 1$ since $H \not\supseteq 2C_4$ and $H \not\supseteq C_3 \uplus K_4$. As $H \not\supseteq 2C_4$, $i(y_2y_4, Q) = 0$. We claim $e(y_4, Q) = 0$. On the contrary, say $e(y_4, Q) \geq 1$. If $y_4b_4 \in E$, then $e(b_4, y_2y_3) = 0$. Moreover, $E(y_2y_3, b_1b_2b_3)$ does not contain two independent edges for otherwise $H \supseteq 2C_4$. Thus $e(y_2y_3, b_1b_2b_3) \leq 3$ and it follows that $e(y_4, Q) = 3$ and $e(y_2y_3, b_1b_2b_3) = 3$. Then either $y_1 \rightarrow (Q; y_2y_3y_4)$ or $y_4 \rightarrow (Q; y_1y_2y_3)$, a contradiction. Hence $y_4b_4 \notin E$. Say w.l.o.g. $y_4b_3 \in E$. Then $y_2b_3 \notin E$ as $i(y_2y_4, Q) = 0$. If $y_3b_4 \in E$ then $y_2b_4 \notin E$ as $e(b_4, y_2y_3y_4) \leq 1$. Moreover, since $[y_3, y_4, b_3, b_4] \supseteq C_4$, $e(y_2, b_1b_2) = 0$ as $H \not\supseteq 2C_4$. Thus $e(y_2, Q) = 0$ and so $e(y_3y_4, Q) \geq 6$. It follows that $y_4 \rightarrow (Q; y_1y_2y_3)$, a contradiction. Hence $y_3b_4 \notin E$. As $i(y_2y_4, Q) = 0$, $e(y_2y_4, Q) \leq 4$. It follows that $e(y_3, Q - b_4) \geq 2$ and so $e(y_3, b_1b_2) \geq 1$. W.l.o.g., say $y_3b_1 \in E$. Then $[y_3, y_4, b_3, b_1] \supseteq C_4$ and so $y_2b_4 \notin E$ as $H \not\supseteq 2C_4$. Thus $e(y_2y_4, Q) = e(y_2y_4, Q - b_4) \leq 3$ as $i(y_2y_4, Q) = 0$. Consequently, $e(y_3, b_1b_2b_3) = 3$ and $e(y_2y_4, b_1b_2b_3) = 3$. As $y_4 \not\rightarrow (Q; y_1y_2y_3)$, we see that $e(y_4, b_1b_2) = 0$. Consequently, $e(y_2, b_1b_2) = 2$. Then $[y_3, y_4, b_3] \cong C_3$ and $[y_1, y_2, b_1, b_2] \cong K_4$, a contradiction. Hence $e(y_4, Q) = 0$. If $y_3b_4 \notin E$ then (a) holds. If $y_3b_4 \in E$ then $y_2b_4 \notin E$. Since $e(y_2, b_1b_2b_3) + e(y_3, Q) \geq 6$, $y_3 \Rightarrow (Q, b_i)$ and $b_iy_2 \in E$ for some $i \in \{1,2,3\}$. Thus $H \supseteq C_3 \uplus K_4$, a contradiction. ∎

Lemma 3.6 *Let P' and P'' be two paths of order 2 and Q a 4-cycle of G such that they are disjoint and $\{P' \cup P'', Q\}$ is optimal. If $e(P' \cup P'', Q) \geq 9$ and $[P', P'', Q] \not\supseteq 2C_4$ then either $[P', P'', Q] \supseteq C_3 \uplus C_4^+$ or $[P', P'', Q] \supseteq P_4 \uplus K_4$.*

Proof. Let $P' = x_1x_2$, $P'' = x_3x_4$, $Q = a_1a_2a_3a_4a_1$ and $H = [P', P'', Q]$. On the contrary, suppose that $H \not\supseteq P_4 \uplus K_4$ and $H \not\supseteq C_3 \uplus C_4^+$. As $e(P' \cup P'', Q) \geq 9$, say w.l.o.g. $e(x_1x_2, Q) \geq 5$. As $e(x_1x_2, Q) = e(x_1x_2, a_1a_2) + e(x_1x_2, a_3a_4)$, say w.l.o.g. $e(x_1x_2, a_1a_2) \geq 3$. Then $[x_1, x_2, a_1, a_2] \supseteq C_4^+$ and so $[x_3, x_4, a_3, a_4] \not\supseteq C_i$ for $i = 3, 4$. Thus $e(x_3x_4, a_3a_4) \leq 1$. If we also have $e(x_1x_2, a_3a_4) \geq 3$, then $e(x_3x_4, a_1a_2) \leq 1$ and so $e(x_1x_2, Q) \geq 7$. W.l.o.g., say $e(x_1x_2, a_1a_2) = 4$. Then $e(x_3x_4, a_3a_4) = 0$ as $H \not\supseteq P_4 \uplus K_4$. Thus $e(x_1x_2, Q) = 8$, $e(x_3x_4, a_1a_2) = 1$ and so $H \supseteq P_4 \uplus K_4$, a contradiction. Hence $e(x_1x_2, a_3a_4) \leq 2$. Similarly, if $e(x_3x_4, a_1a_2) \geq 3$, then $e(x_1x_2, a_3a_4) \leq 1$ and so $e(a_1a_2, P' \cup P'') \geq 7$. Consequently, $e(a_1a_2, x_1x_2) = 4$ or $e(a_1a_2, x_3x_4) = 4$. W.l.o.g., say $e(a_1a_2, x_1x_2) = 4$. Then $e(a_3a_4, x_3x_4) = 0$ as $H \not\supseteq P_4 \uplus K_4$. Thus $e(a_1a_2, P' \cup P'') = 8$, $e(x_1x_2, a_3a_4) = 1$ and so $H \supseteq P_4 \uplus K_4$, a contradiction. Hence $e(x_3x_4, a_1a_2) \leq 2$. As $e(P' \cup P'', Q) \geq 9$, it follows that $e(x_1x_2, a_1a_2) = 4$ and $e(x_3x_4, a_3a_4) = 1$. Thus $H \supseteq P_4 \uplus K_4$, a contradiction. ∎

4 Proofs of Claims 2.1-2.7

Our proof will go along a series of lemmas.

Lemma 4.1 *Let $(T, Q_1, \ldots, Q_{k-1})$ be a feasible chain of G and x the terminal point of $(T, Q_1, \ldots, Q_{k-1})$. Then the following two statements hold:*

(a) For each Q_i, if $e(x, Q_i) \geq 3$ then $x \to Q_i$. Furthermore, for each $u \in V(Q_i)$, if $e(x, Q_i - u) = 3$ then $uu^* \in E$ and if $e(x, Q_i) = 4$ then $\tau(Q_i) = 2$.

(b) For each Q_i, if $e(T, Q_i) \geq 10$ then $\tau(Q_i) \geq 1$ and for some $\{y, z\} \subseteq V(T)$ with $y \neq z$, $y \to Q_i$ and $z \to Q_i$. Moreover, if $\tau(Q_i) = 1$ then there exists $a \in V(Q_i)$ such that $aa^* \in E$ and $N(y, Q_i) = \{a, a^*\}$ for some $y \in V(T)$. Furthermore, if $e(T, Q_i) \geq 11$ then $\tau(Q_i) = 2$ and $y \to Q_i$ for all $y \in V(T)$.

Proof. To see (a), let $u \in V(Q_i)$ be such that $e(x, Q_i - u) = 3$. By (1), $x \overset{na}{\to} (Q_i, u)$. This implies $uu^* \in E$. Thus $x \to Q_i$ and (a) follows.

To see (b), say $Q_i = a_1a_2a_3a_4a_1$ and $T = x_1x_2x_3x_1$. If $\tau(Q_i) = 2$ then $x_r \to Q_i$ for each $x_r \in V(T)$ with $e(x_r, Q_i) \geq 3$ and so the lemma holds. So assume $\tau(Q_i) \leq 1$. As $e(T, Q_i) \geq 10$, $e(x_j, Q_i) = 4$ for some $x_j \in V(T)$. Say w.l.o.g. $e(x_1, Q_i) = 4$. By (1), $x_1 \overset{na}{\to} (Q_i, u)$ and so $uu^* \in E$ for all $u \in I(x_2x_3, Q_i)$. As $i(x_2x_3, Q_i) \geq 2$, say w.l.o.g. with $a_1 \in I(x_2x_3, Q_i)$. Then $a_1a_3 \in E$. As $\tau(Q_i) = 1$, it follows that $I(x_2x_3, Q_i) = \{a_1, a_3\}$. By (1), $x_2 \overset{na}{\to} (Q_i; x_1x_3)$ and $x_3 \overset{na}{\to} (Q_i; x_1x_2)$. This implies that $e(x_2x_3, Q_i) = 6$ with $N(x_r, Q_i) = \{a_1, a_3\}$ for some $r \in \{2, 3\}$. ∎

Lemma 4.2 *There exists no sequence $(P, Q'_1, Q'_2, \ldots, Q'_{k-1})$ of k disjoint subgraphs of G with $P \supseteq 2P_2$ and $Q'_i \cong C_4 (1 \leq i \leq k-1)$ such that $\sum_{i=1}^{k-1} \tau(Q'_i) \geq \sum_{i=1}^{k-1} \tau(Q_i) + 2$.*

Proof. On the contrary, suppose that there exists a sequence $(P, Q'_1, Q'_2, \ldots, Q'_{k-1})$ as described in the lemma such that $\sum_{i=1}^{k-1} \tau(Q'_i) \geq \sum_{i=1}^{k-1} \tau(Q_i) + 2$. Subject to this, we choose $(P, Q'_1, Q'_2, \ldots, Q'_{k-1})$ such that $\sum_{i=1}^{k-1} \tau(Q'_i)$ is maximal. As $G \not\supseteq kC_4$, $[P] \not\supseteq C_4$. By (1), $[P] \not\supseteq C_3$ and so $e([P]) \leq 3$. Thus $e(P, \cup_{i=1}^{k-1} Q'_i) \geq 8k - 6 = 8(k-1) + 2$. This implies that $e(P, Q'_i) \geq 9$ for some $1 \leq i \leq k-1$. Say $i = 1$. By (1), $[P \cup Q'_1] \not\supseteq C_3 \uplus C_4^+$. By Lemma 3.6, $[P, Q'_1] \supseteq P' \uplus Q''$ such that $P' \cong P_4$ and $Q'' \cong K_4$. As $P' \supseteq 2P_2$ and by the maximality of $(P, Q'_1, Q'_2, \ldots, Q'_{k-1})$, $\tau(Q'_1) = 2$. Replacing P and Q'_1 by P' and Q'', we see that either $e(P', Q'') \geq 9$ or $e(P', Q'_j) \geq 9$ for some $j \in \{2, 3, \ldots, k-1\}$. By Lemma 3.5, $[P', Q''] \supseteq C_3 \uplus C_4^+$ or $[P', Q'_j] \supseteq C_3 \uplus C_4^+$, contradicting (1). ∎

Proof of Claim 2.1. On the contrary, suppose that there exists no strong feasible chain in G. Among all the feasible chains of G, we choose $(T, Q_1, \ldots, Q_{k-1})$ such that if u denotes its terminal point then $e(u, Q_1)$ is maximal. As $e(u, G) \geq 2k$, $e(u, Q_1) \geq 3$. If $e(u, Q_1) = 4$, let v and w be two distinct vertices of Q_1. If $e(u, Q_1) = 3$, then

$e(u,Q_i) \le 3$ for all $i \in \{1,\dots,k-1\}$. In this situation, $e(u,Q_i) = 3$ for some $i \in \{2,\dots,k-1\}$ as $e(u,G) \ge 2k$, and then we may assume w.l.o.g. that $e(u,Q_2) = 3$. Then let $v \in V(Q_1)$ and $w \in V(Q_2)$ be such that $e(u,Q_1-v) = 3$ and $e(u,Q_2-w) = 3$. In any case, we define $S = \{u,v,w\}$. By Lemma 4.1, If $e(u,Q_1) = 4$ then $\tau(Q_1) = 2$ and if $e(u,Q_1) = 3$ and $e(u,Q_2) = 3$ then $vv^* \in E$, $ww^* \in E$, $u \Rightarrow (Q_1,v)$ and $u \Rightarrow (Q_2,w)$. Say $T = x_1x_2x_3x_1$ and $R = \{x_1,x_2,x_3\} \cup S$. Let $G' = [u,T,Q_1]$ if $e(u,Q_1) = 4$ and otherwise $G' = [u,T,Q_1,Q_2]$. We shall estimate $e(R,G')$. If $e(u,Q_1) = 4$, then $u \Rightarrow Q_1$ and so $e(y,T) = 0$ for all $y \in V(Q_1)$ for otherwise the claim holds. Thus $e(R,G') = 18$. If $e(u,Q_1) = 3$ and $e(u,Q_2) = 3$, then $e(v,T) = 0$ and $e(w,T) = 0$ for similar reasons. As $u \to Q_1$ and $u \to Q_2$, we see that $[T+y] \not\supseteq C_4$ and so $e(y,T) \le 1$ for all $y \in V(Q_1 \cup Q_2) - \{v,w\}$. Furthermore, by the maximality of $e(u,Q_1)$, we see that if $e(u,Q_1) = 3$ then $e(v,Q_2) \le 3$ and $e(w,Q_1) \le 3$. It follows that if $e(u,Q_1) = 3$ then $e(T,G') \le 12$, $e(S,G') \le 18$ and so $e(R,G') \le 30$. Therefore, if $e(u,Q_1) = 4$ then $e(R,G-V(G')) \ge 12k-18 = 12(k-2)+6$ and if $e(u,Q_1) = 3$ then $e(R,G-V(G')) \ge 12k-30 = 12(k-3)+6$. In any case, there exists Q_r in $G-V(G')$ such that $e(R,Q_r) \ge 13$. Let $u' \in S$ be such that $e(u',Q_r) \ge e(z,Q_r)$ for all $z \in S$. Evidently, we may assume w.l.o.g. $u = u'$. As $e(R,Q_r) \ge 13$, $e(u,Q_r) \ge 1$ and $e(T,Q_r) \ge 1$. Let $Q_r = c_1c_2c_3c_4c_1$. If $e(u,Q_r) = 4$ then $e(c_i,T) = 0$ for all $c_i \in V(Q_r)$ for otherwise the claim holds, a contradiction. Hence $e(u,Q_r) \le 3$.

First, suppose $e(u,Q_r) = 3$. Then $e(S,Q_r) \le 9$ and so $e(T,Q_r) \ge 4$. By Lemma 4.1(a), $u \to Q_r$ and so $e(c_i,T) \le 1$ for all $c_i \in V(T)$ since $[u,Q_r,T] \not\supseteq 2C_4$. Thus $e(c_i,T) = 1$ for all $c_i \in V(T)$. Say w.l.o.g. $e(u,c_1c_2c_3) = 3$. Then $u \Rightarrow (Q_r,c_4)$ and $e(c_4,T) = 1$. Thus the claim holds, a contradiction.

Next, suppose $e(u,Q_r) = 2$. Then $e(S,Q_r) \le 6$ and so $e(T,Q_r) \ge 7$. Assume for the moment that $e(u,c_ic_i^*) = 2$ for some $c_i \in V(Q_r)$. Say w.l.o.g. $e(u,c_1c_3) = 2$. As $[u,Q_r,T] \not\supseteq 2C_4$, $u \not\to (Q_r,c_j;V(T))$ and so $e(c_j,T) \le 1$ for $j \in \{2,4\}$. As $e(T,Q_r) \ge 7$, either $e(c_1c_2,T) = 4$ or $e(c_3c_4,T) = 4$. W.l.o.g., say $e(c_1,T) = 3$ and $e(c_2,T) = 1$. Then $u \not\Rightarrow (Q_r,c_2)$ for otherwise the claim holds. This implies $c_2c_4 \in E$. Thus $[c_2,c_3,c_4] \cong C_3$, $e(u,c_2c_3c_4) = 1$ and $[c_1,T] \cong K_4$, i.e., the claim holds, a contradiction. This argument shows that $\tau(Q_r) \le 1$ for otherwise we may choose a 4-cycle from $[Q_r]$ such that u is adjacent to two non consecutive vertices of this 4-cycle and repeat the above argument to obtain a contradiction. W.l.o.g., say $e(u,c_1c_2) = 2$. Assume for the moment that $e(c_i,T) \ge 2$ for some $i \in \{3,4\}$. Say w.l.o.g. $e(c_4,T) \ge 2$. Then $[c_4,T] \ge Q_r$, $[u,c_1,c_2] \cong C_3$ and $e(c_3,uc_1c_2) = 1$. Therefore the claim holds, a contradiction. Hence $e(c_3,T) \le 1$ and $e(c_4,T) \le 1$. Thus $e(c_1c_2,T) \ge 5$. Let $j \in \{1,2\}$ be such that $e(c_j,T) = 3$. Then $[c_j,T] \cong K_4$ and $[u,Q_r-c_j] \supseteq 2P_2$. By Lemma 4.2, $\tau(Q_r) \ne 0$. W.l.o.g., say $c_1c_3 \in E$. Then $u \Rightarrow (Q_r,c_4)$. Since the claim does not hold, $e(c_4,T) = 0$. It follows that $(c_1c_2,T) = 6$ and $e(c_3,T) = 1$. Thus $[c_2,T] > Q_r$ and $[c_1,c_3,c_4] \cong C_3$, contradicting (1).

Finally, $e(u,Q_r)=1$. Then $e(S,Q_r)\le 3$ and so $e(T,Q_r)\ge 10$. By Lemma 4.1(b), $\tau(Q_r)\ge 1$. Moreover, if $\tau(Q_r)=1$, we may assume that $c_1c_3\in E$ and $N(x_i,Q_r)=\{c_1,c_3\}$ for some $x_i\in V(T)$. W.l.o.g., say $e(u,c_1c_2c_3)=1$. Then $T+c_4\supseteq C_4^+$ and so the claim holds, a contradiction. Hence $\tau(Q_r)=2$. W.l.o.g., say $uc_1\in E$. Then $e(u,c_1c_ic_j)=1$, $[c_1,c_i,c_j]\cong C_3$ and so $T+c_t\not\supseteq K_4$ for each permutation (i,j,t) of $\{2,3,4\}$. This implies that $e(c_i,T)\le 2$ for $i\in\{2,3,4\}$ and so $e(T,Q_r)\le 9$, a contradiction. This proves Claim 2.1. ∎

By Claim 2.1, we choose a strong feasible chain $\sigma=(x_0x_1,T,Q_1,\ldots,Q_{k-1})$ with $x_1\in V(T)$. Let $T=x_1x_2x_3x_1$, $F=x_0x_1x_2x_3x_1$ and $\mathcal{Q}=\{Q_1,\ldots,Q_{k-1}\}$. Set $G_i=[F,\cup_{r=1}^{i}Q_r]$ and $H_i=G-V(G_i)$ for each $i\in\{1,\ldots,k-1\}$. Clearly, $G_i\not\supseteq(i+1)C_4$ for each $i\in\{1,\ldots,k-1\}$. A *terminal point* of G is a terminal point of some feasible chain of G. Let $\mathcal{T}$ be the set of all the terminal points of G. The following Lemma 4.3 and Lemma 4.4 are the initial elimination process for the proofs of Claims 2.2-2.5.

Lemma 4.3 *Let $Q\in\mathcal{Q}$. If $e(F,Q)\ge 9$, $e(x_0,Q)>0$ and $[F,Q]\not\supseteq 2C_4$, then there exist a labelling $F=z_0z_1z_2z_3z_1$ and a 4-cycle $a_1a_2a_3a_4a_1$ in $[Q]$ such that one of the following statements (3) to (8) holds:*

$$N(z_0,Q)=\{a_1\},N(z_2,Q)=\{a_1,a_4\},N(z_3,Q)=\{a_1,a_2\},e(z_1,Q)=4,a_1a_3\in E,a_2a_4\notin E;\quad(3)$$
$$N(z_0z_2z_3,Q)\subseteq\{a_1,a_3\},3\le e(z_1,Q)\le 4,a_1a_3\in E;\quad(4)$$
$$N(z_0z_1,Q)\subseteq\{a_1,a_3\},N(z_2,Q)\subseteq\{a_1,a_4,a_3\},N(z_3,Q)\subseteq\{a_1,a_2,a_3\},a_1a_3\in E,a_2a_4\notin E;\quad(5)$$
$$N(z_0,Q)\subseteq\{a_1,a_2\},N(z_2,Q)\subseteq\{a_1,a_2,a_3\},N(z_3,Q)\subseteq\{a_1\},a_1a_3\in E,a_2a_4\notin E;\quad(6)$$
$$N(z_0,Q)=\{a_1\},N(z_1,Q)=N(z_2,Q)=\{a_1,a_2,a_3\},N(z_3,Q)=\{a_1,a_3\},a_1a_3\in E,a_2a_4\notin E;\quad(7)$$
$$N(z_0,Q)=\{a_1\},e(z_1z_2,Q)=8,e(z_3,Q)=0,a_1a_3\in E.\quad(8)$$

In addition, if (3) or (8) holds then $[T,Q,v]\supseteq 2C_4$ for each $v\in V(G)-V(F\cup Q)$ with $e(v,Q)\ge 2$.

Proof. The last statement is obvious since $v\to(Q,a)$ for some $a\in V(Q)$ with $e(a,T)\ge 2$. We proceed to prove one of (3) to (8) to be true. Let $H=[F,Q]$, $F=z_0z_1z_2z_3z_1$ and $Q=a_1a_2a_3a_4a_1$. As $H\not\supseteq 2C_4$, $z_0\not\to(Q;V(T))$. As $e(F,Q)\ge 9$, $e(u,T)\ge 2$ for some $u\in V(Q)$. Then $z_0\not\to Q$ and so $e(z_0,Q)\le 2$ by Lemma 4.1(a). We now divide the proof into the following two cases.

Case 1. $e(z_0,Q)=2$.

In this case, $e(T,Q)\ge 7$. First, suppose that $e(z_0,a_1a_3)=2$ or $e(z_0,a_2a_4)=2$. W.l.o.g., say the former holds. Then $z_0\to(Q,a_i)$ for $i\in\{2,4\}$. As $H\not\supseteq 2C_4$, $e(a_2,T)\le 1$ and $e(a_4,T)\le 1$. Thus $e(a_1a_3,T)\ge 5$. W.l.o.g., say $e(a_1,T)=3$ and $e(a_3,T)\ge 2$. As $H\not\supseteq 2C_4$, $z_2\not\to(Q;z_0z_1z_3)$ and so $e(z_2,a_2a_4)\le 1$. Similarly, $e(z_3,a_2a_4)\le 1$. Assume that $e(z_2,Q)=3$ or $e(z_3,Q)=3$. W.l.o.g., say

$e(z_2, a_1a_4a_3) = 3$. Then $e(a_4, z_1z_3) = 0$. As $z_2 \not\to (Q; z_0z_1z_3)$, $a_2a_4 \notin E$. As $[z_0, z_1, z_3, a_1] \cong C_4^+$ and $[a_3, a_4, z_2] \cong C_3$, $H \supseteq C_3 \uplus C_4^+$ and so $a_1a_3 \in E$ by (1). If $z_1a_2 \notin E$, then (5) holds. If $z_1a_2 \in E$, then $H \supseteq 2C_4 = \{z_1a_2a_3z_0z_1, z_2a_4a_1z_3z_2\}$, a contradiction. Next, assume that $e(z_2, Q) \leq 2$ and $e(z_3, Q) \leq 2$. We claim $e(z_2z_3, a_2a_4) = 0$. If this is false, say w.l.o.g. $z_2a_4 \in E$. Then $e(a_4, z_1z_3) = 0$. As $e(T, Q) \geq 7$, $e(z_1z_3, a_1a_2a_3) \geq 5$. It follows that $e(a_1a_3, z_1z_3) = 4$ and $e(a_2, z_1z_3) = 1$. As $e(z_3, Q) < 3$, $a_2z_1 \in E$. Thus $H \supseteq 2C_4 = \{z_1a_2a_3z_0z_1, z_2a_4a_1z_3z_2\}$, a contradiction. Hence $e(z_2z_3, a_2a_4) = 0$. It remains to show that $a_1a_3 \in E$ and so (4) holds. Clearly, if $a_2a_4 \in E$, then $[a_2, a_3, a_4] \cong C_3$, $H \supseteq C_3 \uplus K_4$ and so $a_1a_3 \in E$ by (1). On the contrary, say $a_1a_3 \notin E$. Then $a_2a_4 \notin E$ and so $\tau(Q) = 0$. If $e(z_1, a_2a_4) = 2$, then $z_1 \xrightarrow{a} (Q, a_3)$. By (1), $[z_2, z_3, a_3] \not\supseteq C_3$ and so $e(a_3, z_2z_3) \leq 1$. As $e(T, Q) \geq 7$, it follows that $e(z_1, Q) = 4$. Thus $z_1 \xrightarrow{a} (Q, a_1)$ and $[a_1, z_2, z_3] \supseteq C_3$, contradicting (1). If $e(z_1, a_2a_4) \leq 1$, say $z_1a_2 \notin E$. Then $e(z_1, a_1a_4a_3) = 3$ and $e(z_2z_3, a_1a_3) = 4$. Consequently, $[z_0, z_1, a_4, a_1] \supseteq C_4^+$ and $[z_2, z_3, a_3] \supseteq C_3$, contradicting (1).

Next, suppose $e(z_0, a_ia_{i+1}) = 2$ for some $a_i \in V(Q)$. Say w.l.o.g. $e(z_0, a_1a_2) = 2$. We may assume that $\tau(Q) \leq 1$ for otherwise we choose a 4-cycle Q' from $[Q]$ such that a_1 and a_2 are not consecutive on Q' and repeat the above argument. Thus $H \not\supseteq C_3 \uplus K_4$ by (1). As $[z_0, a_1, a_2] \cong C_3$ and $H \not\supseteq C_3 \uplus K_4$, we see that $e(a_4, T) \leq 2$ and $e(a_3, T) \leq 2$. If $e(a_3, T) = 2$ or $e(a_4, T) = 2$, then $H \supseteq C_3 \uplus C_4^+$ and so $\tau(Q) \geq 1$ by (1). If $e(a_3, T) \leq 1$ and $e(a_4, T) \leq 1$, then $e(a_1, T) = 3$ or $e(a_2, T) = 3$ and so $H \supseteq 2P_2 \uplus K_4$. Then by Lemma 4.2, $\tau(Q) \neq 0$. We conclude that $\tau(Q) = 1$. W.l.o.g., say $a_1a_3 \in E$. Then $[a_1, a_4, a_3] \cong C_3$ and $z_0 \to (Q, a_4)$. Thus $e(a_2, T) \leq 2$ and $e(a_4, T) \leq 1$ as $H \not\supseteq C_3 \uplus K_4$ and $H \not\supseteq 2C_4$. We shall prove that (6) holds. We claim $e(a_4, z_2z_3) = 0$. If false, say $a_4z_2 \in E$. Then $e(a_4, z_1z_3) = 0$. If $z_3a_3 \in E$ then $[z_3, a_3, a_4, z_2] \supseteq C_4$ and so $e(z_1, a_1a_2) = 0$ as $H \not\supseteq 2C_4$. Similarly, if $z_3a_1 \in E$ then $z_1a_3 \notin E$. This implies that $e(z_1z_3, a_1a_2a_3) \leq 4$ and if $e(z_1z_3, a_1a_2a_3) = 4$ then $e(a_2, z_1z_3) = 2$. As $e(z_2, Q) \geq 7 - e(z_1z_3, Q) \geq 3$, we see that $e(z_2, Q - a_2) \geq 2$ and so $z_2 \to (Q, a_2)$. As $H \not\supseteq 2C_4$, $z_2 \not\to (Q, a_2; z_0z_1z_3)$. Thus $a_2z_3 \notin E$. We conclude that $e(z_1z_3, a_1a_2a_3) \leq 3$. It follows that $e(z_2, Q) = 4$ and $e(z_1z_3, a_1a_2a_3) = 3$. As $z_2 \xrightarrow{a} (Q, a_2)$, $[z_0, z_1, a_2] \not\supseteq C_3$ by (1) and so $a_2z_1 \notin E$. Thus $e(a_2, z_1z_3) = 0$. As $H \not\supseteq 2C_4$, $z_2 \not\to (Q, a_1; z_0z_1z_3)$ and so $a_1z_3 \notin E$. Thus $e(a_3, z_1z_3) = 2$ as $e(z_1z_3, a_1a_2a_3) = 3$, and so $e(a_3, T) = 3$, a contradiction. Hence $e(a_4, z_2z_3) = 0$. Next, we claim $e(a_3, z_2z_3) \leq 1$. If false, say $e(a_3, z_2z_3) = 2$. Then $z_1a_3 \notin E$ as $e(a_3, T) \leq 2$. As $[a_3, z_2, z_3] \cong C_3$ and $H \not\supseteq C_3 \uplus K_4$, $e(z_1, a_1a_2) \leq 1$. Thus $e(z_1, Q) \leq 2$ and so $e(z_2z_3, a_1a_2) \geq 7 - 2 - 2 = 3$. Then $\{a_1z_i, a_2z_j\} \subseteq E$ for some $\{i, j\} = \{2, 3\}$. Thus $z_i \to (Q, a_2; z_0z_1z_j)$, i.e., $H \supseteq 2C_4$, a contradiction. Hence $e(a_3, z_2z_3) \leq 1$. As $e(z_2z_3, Q) \geq 9 - e(z_0z_1, Q) \geq 3$, we may assume w.l.o.g. that $e(z_2, Q) \geq 2$. If $N(z_3, Q) \subseteq \{a_1\}$ then (6) holds. So suppose $e(z_3, a_2a_3) \geq 1$. First, assume $z_3a_2 \in E$. Then $e(z_2, a_1a_3) \leq 1$ as $z_2 \not\to (Q, a_2; z_0z_1z_3)$. Thus $e(z_2, a_1a_3) = 1$ and $z_2a_2 \in E$. Then $z_1a_2 \notin E$ as $e(a_2, T) \leq 2$. As $z_3 \not\to (Q, a_2; z_0z_1z_2)$, $e(z_3, a_1a_3) \leq 1$.

It follows that $e(z_1, a_1a_4a_3) \geq 7-2-2 = 3$. Thus $z_1 \overset{a}{\to} (Q, a_2)$ and $[a_2, z_2, z_3] \cong C_3$, contradicting (1). Hence $z_3a_2 \notin E$. Finally, assume $z_3a_3 \in E$. Then $z_2a_3 \notin E$ as $e(a_3, z_2z_3) \leq 1$. Hence $e(z_2, a_1a_2) = 2$. Then $z_3a_1 \notin E$ as $z_3 \not\to (Q, a_2; z_0z_1z_2)$. Thus $e(z_2z_3, Q) = 3$ and so $e(z_1, Q) = 4$. Then $H \supseteq 2C_4 = \{z_0z_1a_4a_1z_0, z_2a_2a_3z_3z_2\}$, a contradiction.

Case 2. $e(z_0, Q) = 1$.

Then $e(T, Q) \geq 8$. Say $z_0a_1 \in E$. If $e(z_3, Q) = 0$ or $e(z_2, Q) = 0$, we assume $e(z_3, Q) = 0$. Then $e(z_1z_2, Q) = 8$ and $[z_0, a_1, z_1] \cong C_3$. By (1), $z_2 \overset{na}{\to} (Q, a_1)$ and so $a_1a_3 \in E$. Thus (8) holds. Hence we may assume $e(z_3, Q) \geq 1$ and $e(z_2, Q) \geq 1$. Suppose $e(z_3, Q) = 1$ or $e(z_2, Q) = 1$. Say the former holds. If $z_3a_1 \in E$, then $e(z_2, a_2a_4) \leq 1$ as $z_2 \not\to (Q, a_1; z_0z_1z_3)$. Thus $e(z_1, Q) = 4$ and $e(z_2, Q) = 3$. W.l.o.g., say $e(z_2, a_1a_2a_3) = 3$. Then $a_2a_4 \notin E$ as $z_2 \not\to (Q, a_1; z_0z_1z_3)$. As $[a_1, z_2, z_3] \cong C_3$ and by (1), $z_1 \overset{na}{\to} (Q, a_1)$ and so $a_1a_3 \in E$. Thus (6) holds. If $z_3a_3 \in E$, then it is easy to see that $E(z_1z_2, a_2a_4)$ does not contain two independent edges for otherwise $H \supseteq 2C_4$. Consequently, $e(z_1z_2, a_2a_4) \leq 2$ and so $e(T, Q) \leq 7$, a contradiction. Hence $e(z_3, a_2a_4) = 1$. Say w.l.o.g. $z_3a_2 \in E$. As above, if $\tau(Q) = 2$ then $E(z_1z_2, a_3a_4)$ does not contain two independent edges since $H \not\supseteq 2C_4$ and so $e(T, Q) \leq 7$, a contradiction. Hence $\tau(Q) \leq 1$. If $z_2a_3 \in E$ then $z_1a_4 \notin E$ as $H \not\supseteq 2C_4$. Consequently, $e(z_2, Q) = 4$ and $e(z_1, a_1a_2a_3) = 3$. Clearly, $[z_1, z_0, a_1] \cong C_3$ and $[z_1, z_3, a_2] \cong C_3$. By (1), $z_2 \overset{na}{\to} (Q, a_1)$ and $z_2 \overset{na}{\to} (Q, a_2)$, which implies that $\tau(Q) = 2$, a contradiction. Hence $z_2a_3 \notin E$. It follows that $e(z_2, a_2a_1a_4) = 3$ and $e(z_1, Q) = 4$. Then $[a_2, z_2, z_3] \supseteq C_3$. By (1) $z_1 \overset{na}{\to} (Q, a_2)$ and so $a_2a_4 \in E$. By exchanging the subscripts of a_1 with a_2 and a_3 with a_4, we see that (6) holds. Therefore we may assume below that $e(z_i, Q) \geq 2$ for $i \in \{2, 3\}$.

First, suppose that either $e(z_3, Q) = 2$ or $e(z_2, Q) = 2$. Say the former holds. Then $e(z_1z_2, Q) \geq 6$. Assume for the moment $e(z_3, a_2a_4) = 2$. Then $z_2a_1 \notin E$ as $z_3 \not\to (Q, a_1; z_0z_1z_2)$. Thus $e(z_2, Q) \leq 3$ and so $e(z_1, Q) \geq 3$. Hence $e(z_1, a_2a_4) \geq 1$. W.l.o.g., say $z_1a_2 \in E$. Then $[z_0, z_1, a_2, a_1] \supseteq C_4$. Thus $z_2a_3 \notin E$ as $H \not\supseteq 2C_4$. It follows that $e(z_1, Q) = 4$ and $e(z_2, a_2a_4) = 2$. Clearly, $[a_2, z_2, z_3] \cong C_3$. By (1), $z_1 \overset{na}{\to} (Q, a_2)$ and so $a_2a_4 \in E$. Then $[z_0, z_1, a_1] \cong C_3$ and $[a_2, a_4, z_2, z_3] \cong K_4$. By (1), $\tau(Q) = 2$. Then $[z_0, z_1, a_3, a_1] \supseteq C_4$ and so $H \supseteq 2C_4$, a contradiction. Hence $e(z_3, a_2a_4) \neq 2$. Next, assume $e(z_3, a_1a_3) = 2$. As $z_2 \not\to (Q, a_1; z_0z_1z_3)$, $e(z_2, a_2a_4) \leq 1$. Hence $e(z_2, Q) \leq 3$ and so $e(z_1, Q) \geq 3$. If $e(z_2, Q) = 3$, we may assume $e(z_2, a_1a_2a_3) = 3$. Then $[a_2, a_3, z_2, z_3] \supseteq C_4$ and so $z_1a_4 \notin E$ as $H \not\supseteq 2C_4$. Consequently, $e(z_1, a_1a_2a_3) = 3$. As $z_2 \not\to (Q, a_1; z_0z_1z_3)$, $a_2a_4 \notin E$. Clearly, $[z_0, z_1, a_2, a_1] \cong C_4^+$ and $[a_3, z_2, z_3] \cong C_3$. By (1), $\tau(Q) = 1$, i.e., $a_1a_3 \in E$, and so (7) holds. Hence we may assume $e(z_2, Q) \leq 2$. It follows that $e(z_2, Q) = 2$ and $e(z_1, Q) = 4$. As $H \not\supseteq 2C_4$, we readily see $e(z_2, a_2a_4) = 0$. Thus $e(z_2, a_1a_3) = 2$. As $[a_1, z_2, z_3] \cong C_3$, $z_1 \overset{na}{\to} (Q, a_1)$ by (1). Thus $a_1a_3 \in E$ and so (4) holds. Next, assume that $e(z_3, a_4a_3) = 2$ or $e(z_3, a_2a_3) = 2$. Say the former holds. If $z_1a_2 \in E$ then $[z_0, z_1, a_2, a_1] \supseteq C_4$ and so $e(z_2, a_3a_4) = 0$ as $H \not\supseteq 2C_4$. Consequently, $e(z_1, Q) = 4$, $e(z_2, a_1a_2) = 2$ and clearly, $H \supseteq 2C_4$, a contradiction. Hence $z_1a_2 \notin E$. Thus $e(z_1, Q) \leq 3$ and so $e(z_2, Q) \geq 3$. If $z_2a_2 \notin E$ then $e(z_1z_2, a_1a_4a_3) = 6$. If $z_2a_2 \in E$, then $z_1a_4 \notin E$ because $[z_2, a_2, a_3, z_3] \supseteq C_4$ and $H \not\supseteq 2C_4$. Consequently, $e(z_2, Q) = 4$ and $e(z_1, a_1a_3) = 2$. In either situation, $[a_1, z_0, z_1] \cong C_3$ and $[z_2, z_3, a_3, a_4] \cong K_4$. By (1), $\tau(Q) = 2$ and so $z_3 \to (Q, a_1; z_0z_1z_2)$, a contradiction. Finally, assume that $e(z_3, a_1a_2) = 2$ or $e(z_3, a_1a_4) = 2$. Say the former holds. As $z_2 \not\to (Q, a_1; z_0z_1z_3)$, $e(z_2, a_2a_4) \leq 1$. Thus $e(z_2, Q) \leq 3$ and so $e(z_1, Q) \geq 3$. We claim that $z_2a_3 \notin E$. If false, then $[z_3, z_2, a_3, a_2] \supseteq C_4$ and so $z_1a_4 \notin E$ as $H \not\supseteq 2C_4$. Thus $e(z_1, a_1a_2a_3) = 3$, $e(z_2, a_1a_3) = 2$ and $e(z_2, a_2a_4) = 1$. As $z_2 \not\to (Q, a_1; z_0z_1z_3)$, $a_2a_4 \notin E$. As $[a_2, z_1, z_3] \cong C_3$, $z_2 \overset{na}{\to} (Q, a_2)$ by (1) and this implies that $z_2a_4 \notin E$. Thus $z_2a_2 \in E$ and so $e(a_2, T) = 3$, i.e., $\tau(a_2z_1z_2z_3a_2) = 2 > \tau(Q)$. By (1), $[a_1, a_4, a_3] \not\supseteq C_3$ and so $a_1a_3 \notin E$. Thus $\tau(Q) = 0$. But, as $[z_0, a_1, a_4, a_3] \supseteq 2P_2$, we obtain a contradiction with Lemma 4.2. Hence $z_2a_3 \notin E$. Thus $z_2a_1 \in E$, $e(z_2, a_2a_4) = 1$ and $e(z_1, Q) = 4$. If $z_2a_2 \in E$, then $[a_1, a_2, z_2, z_3] \cong K_4$, $[z_1, a_3, a_4] \cong C_3$ and so $\tau(Q) = 2$ by (1). Consequently, (4) holds by exchanging the subscripts of a_2 with a_3. Hence assume $z_2a_2 \notin E$ and $z_2a_4 \in E$. As $[a_1, z_2, z_3] \cong C_3$, $z_1 \overset{na}{\to} (Q, a_1)$ by (1) and so $a_1a_3 \in E$. Then $a_2a_4 \notin E$ for otherwise $H \supseteq 2C_4 = \{z_2z_3a_2a_4z_2, z_0z_1a_3a_1z_0\}$. Then (3) holds.

Finally, suppose that $e(z_2, Q) \geq 3$ and $e(z_3, Q) \geq 3$. First, assume that either $e(z_2, a_2a_4) = 2$ or $e(z_3, a_2a_4) = 2$. Say the former holds. Then $z_3a_1 \notin E$ as $z_2 \not\to (Q, a_1; z_0z_1z_3)$. Thus $e(z_3, a_2a_3a_4) = 3$. Then $z_2a_1 \notin E$ as $z_3 \not\to (Q, a_1; z_0z_1z_2)$ and so $e(z_2, a_2a_3a_4) = 3$. Thus $e(z_1, a_2a_4) = 0$ as $H \not\supseteq 2C_4$. Hence $e(z_1, a_1a_3) = 2$. Obviously, $H \supseteq C_3 \uplus K_4$. Thus $\tau(Q) = 2$ by (1) and so $H \supseteq 2C_4$, a contradiction. Hence $e(z_2, a_2a_4) \leq 1$ and $e(z_3, a_2a_4) \leq 1$. Thus $e(z_2, Q) = e(z_3, Q) = 3$ and $e(z_1, Q) \geq 2$. W.l.o.g., say $e(z_2, a_1a_4a_3) = 3$. If $z_3a_4 \in E$ then $e(z_3, a_1a_4a_3) = 3$. Thus $z_1a_2 \notin E$ and $a_2a_4 \notin E$ as $H \not\supseteq 2C_4$. Since $\tau(Q) \leq 1$ and $[z_2, z_3, a_3, a_4] \supseteq K_4$, $[a_1, z_0, z_1] \not\supseteq C_3$ by (1) and so $z_1a_1 \notin E$. Thus $e(z_1, a_3a_4) = 2$. As $[a_2, a_1, z_0, z_1] \supseteq 2P_2$ and by Lemma 4.2, $\tau(Q) \neq 0$ and so $a_1a_3 \in E$. Thus $[a_1, a_2, a_3] \cong C_3$, $[T, a_4] \cong K_4$ and so $\tau(Q) = 2$ by (1), a contradiction. Therefore $z_3a_4 \notin E$ and so $e(z_3, a_1a_2a_3) = 3$. Then we see that $e(z_1, a_2a_4) = 0$ and $a_2a_4 \notin E$ as $H \not\supseteq 2C_4$. Thus $e(z_1, a_1a_3) = 2$. Since $[z_0, z_1, z_3, a_1] \cong C_4^+$ and $[a_3, a_4, z_2] \cong C_3$, we obtain $\tau(Q) = 1$ by (1) and so $a_1a_3 \in E$. Thus (5) holds. ∎

Lemma 4.4 *Let $Q \in \mathcal{Q}$. If $e(F - x_1, Q) \geq 7$ with $e(x_0, Q) \geq 1$ then there exist two labellings $F = z_0z_1z_2z_3z_1$ and $Q = u_1u_2u_3u_4u_1$ such that one of the following statements* (9) *to* (14) *holds:*

$$e(z_0, Q) = 1, N(z_2, Q) = N(z_3, Q) = \{u_2, u_3, u_4\}; \tag{9}$$

$$e(z_0,Q)=4, \{u_2,u_3,u_4\}\subseteq N(z_2,Q), e(z_3,Q)=0, \tau(Q)=2; \tag{10}$$
$$N(z_0,Q)=\{u_1,u_2,u_3\}, e(z_2,Q)=4, e(z_3,Q)=0, u_2u_4\in E; \tag{11}$$
$$N(z_0,Q)=N(z_2,Q)=\{u_1,u_2,u_3\}, N(z_3,Q)=\{u_4\}, u_2u_4\in E; \tag{12}$$
$$N(z_0,Q)=\{u_1\}, N(z_2,Q)=\{u_1,u_4,u_3\}, N(z_3,Q)=\{u_1,u_2,u_3\}, u_2u_4\notin E; \tag{13}$$
$$N(z_0,Q)=\{u_1,u_3\}, N(z_2,Q)=\{u_1,u_4,u_3\}, N(z_3,Q)=\{u_1,u_3\}, u_2u_4\notin E. \tag{14}$$

Moreover, if one of (10) *to* (12) *holds, then* $z_2\to Q$ *and* $v\to(Q;z_0z_1z_2)$ *for each* $v\in V(G)-V(F\cup Q)$ *with* $e(v,Q)\ge 2$.

Proof. The last statement is an easy observation. We claim that there exist two labellings $F=z_0z_1z_2z_3z_1$ and $Q=u_1u_2u_3u_4u_1$ such that either one of (9) to (14) holds or one of (15) to (20) holds:

$$N(z_0,Q)=\{u_1\}, e(z_2,Q)=4, N(z_3,Q)=\{u_2,u_3\}, u_1u_3\in E, u_2u_4\notin E; \tag{15}$$
$$N(z_0,Q)=\{u_1,u_3\}, N(z_2,Q)=\{u_1,u_4,u_3\}, N(z_3,Q)=\{u_1,u_2,u_3\}, u_2u_4\notin E; \tag{16}$$
$$N(z_0,Q)=\{u_1,u_3\}, N(z_2,Q)=\{u_1,u_4,u_3\}, N(z_3,Q)=\{u_1,u_2\}, u_2u_4\notin E; \tag{17}$$
$$N(z_0,Q)=\{u_1,u_2\}, e(z_2,Q)=4, N(z_3,Q)=\{u_3\}, u_1u_3\in E, u_2u_4\notin E; \tag{18}$$
$$N(z_0,Q)=\{u_1,u_2\}, N(z_2,Q)=\{u_1,u_2,u_3\}, N(z_3,Q)=\{u_1,u_4\}, \tau(Q)=0; \tag{19}$$
$$N(z_0,Q)=\{u_1,u_2\}, N(z_2,Q)=\{u_1,u_4,u_3\}, N(z_3,Q)=\{u_1,u_3\}, u_1u_3\in E, u_2u_4\notin E. \tag{20}$$

To see these, say w.l.o.g. $Q=Q_1=u_1u_2u_3u_4u_1$. Say $F=z_0z_1z_2z_3z_1$. Suppose $e(z_0,Q_1)\ge 3$. Say $e(z_0,u_1u_2u_3)=3$. By Lemma 4.1(a), $u_2u_4\in E$ and $z_0\to Q_1$. As $G_1\not\supseteq 2C_4$, $e(u_i,z_2z_3)\le 1$ for each $u_i\in V(Q_1)$. If $e(z_0,Q_1)=4$ then $\tau(Q_1)=2$ and consequently, $e(z_2,Q_1)=0$ or $e(z_3,Q_1)=0$ as $G_1\not\supseteq 2C_4$. Say w.l.o.g. $e(z_3,Q_1)=0$ and so (10) holds. If $e(z_0,Q_1)=3$ then $e(u_i,z_2z_3)=1$ for all $u_i\in V(Q_1)$. If $e(z_3,Q_1)=0$ or $e(z_2,Q_1)=0$, say w.l.o.g. $e(z_3,Q_1)=0$, then (11) holds. Hence we may assume w.l.o.g. that $z_3u_4\in E$ and $e(z_2,u_1u_2u_3)\ge 1$. Then $z_3u_2\notin E$ as $z_3\not\to(Q_1;z_0z_1z_2)$. Hence $z_2u_2\in E$. For the same reason, $e(z_3,u_1u_3)=0$ and so $e(z_2,u_1u_3)=2$. Thus (12) holds. Next, suppose $e(z_0,Q_1)=1$. Then $e(z_2z_3,Q_1)\ge 6$. Say $z_0u_1\in E$. Assume $e(z_i,u_2u_4)=2$ for some $i\in\{2,3\}$. Say w.l.o.g. $e(z_2,u_2u_4)=2$. Then $z_3u_1\notin E$ as $z_2\not\to(Q_1,u_1;z_0z_1z_3)$. Similarly, if $e(z_3,u_2u_4)=2$ then $z_2u_1\notin E$, and consequently, $e(z_2z_3,u_2u_3u_4)=6$. Thus (9) holds. If $e(z_3,u_2u_4)\le 1$ then $e(z_3,Q_1)=2$, $e(z_2,Q_1)=4$ and we may assume w.l.o.g. that $e(z_3,u_2u_3)=2$. Then $u_2u_4\notin E$ as $z_3\not\to(Q_1,u_1;z_0z_1z_2)$. Clearly, $[z_2,z_3,u_2,u_3]\supseteq K_4$ and so $G_1\supseteq P_4\uplus K_4$. By Lemma 4.2, $\tau(Q_1)\ne 0$ and so $u_1u_3\in E$. Thus (15) holds. If $e(z_i,u_2u_4)\le 1$ for $i\in\{2,3\}$ then (13) holds or $N(z_2,Q_1)=N(z_3,Q_1)$. If the latter holds then (9) holds (if necessary, exchanging the subscripts of some u_i's).

Therefore we may assume $e(z_0,Q_1)=2$. Then $e(z_2z_3,Q_1)\ge 5$. First, suppose $N(z_0,Q_1)=\{u_i,u_{i+2}\}$ for some $i\in\{1,2\}$. Say w.l.o.g. $e(z_0,u_1u_3)=2$.

Then $e(u_2, z_2z_3) \leq 1$ and $e(u_4, z_2z_3) \leq 1$ as $G_1 \not\supseteq 2C_4$. Then $e(z_i, u_1u_3) = 2$ and $e(z_i, u_2u_4) = 1$ for some $i \in \{2, 3\}$. W.l.o.g., say $e(z_2, u_1u_4u_3) = 3$. As $e(z_3, u_1u_3) \geq 1$ and $z_2 \not\rightarrow (Q_1; z_0z_1z_3)$, $u_2u_4 \notin E$. Hence one of (14), (16) and (17) holds. Next, suppose $N(z_0, Q_1) = \{u_i, u_{i+1}\}$ for some $i \in \{1, 2, 3, 4\}$. W.l.o.g., say $e(z_0, u_1u_2) = 2$ and $e(z_2, Q_1) \geq e(z_3, Q_1)$. If $e(z_2, Q_1) = 4$, then $e(z_3, u_1u_2) = 0$ as $G_1 \not\supseteq 2C_4$. Then $e(z_3, u_3u_4) \geq 1$ and so $G_1 \supseteq C_3 \uplus C_4^+$. Thus $\tau(Q_1) \geq 1$ by (1). Say w.l.o.g. $u_1u_3 \in E$. Then $z_3u_4 \notin E$ and $u_2u_4 \notin E$ as $z_0 \not\rightarrow (Q_1; z_2z_1z_3)$. Thus $z_3u_3 \in E$ and so (18) holds. Hence we may assume $e(z_2, Q_1) = 3$. Then $\{u_1, u_2\} \subseteq N(z_2)$ or $\{u_3, u_4\} \subseteq N(z_2)$. First, assume the former holds. As $e(z_2, u_3u_4) = 1$, say w.l.o.g. $z_2u_3 \in E$. Then $z_3u_2 \notin E$ and $e(z_3, u_1u_3) \leq 1$ as $G_1 \not\supseteq 2C_4$. Thus $z_3u_4 \in E$ and $e(z_3, u_1u_3) = 1$. If $z_3u_3 \in E$ then $[z_2, z_3, u_3, u_4] \supseteq C_4^+$ and so $G_1 \supseteq C_3 \uplus C_4^+$. Thus $\tau(Q_1) \geq 1$ by (1) and consequently, $z_3 \rightarrow (Q_1; z_0z_1z_2)$, a contradiction. Hence $z_3u_3 \notin E$ and so $z_3u_1 \in E$. Then $\tau(Q_1) = 0$ as $G_1 \not\supseteq 2C_4$ and so (19) holds. Therefore we may assume $\{u_3, u_4\} \subseteq N(z_2)$. As $e(z_2, Q_1) = 3$, say w.l.o.g. $e(z_2, u_1u_4u_3) = 3$. Then $z_3u_2 \notin E$ as $z_2 \not\rightarrow (Q_1, u_2; z_0z_1z_3)$. Thus $e(z_3, u_1u_3) \geq 1$. Then $u_2u_4 \notin E$ for otherwise either $z_0 \rightarrow (Q_1; z_2z_1z_3)$ or $z_2 \rightarrow (Q_1; z_0z_1z_3)$. If $z_3u_4 \in E$ then $G_1 \supseteq C_3 \uplus C_4^+$ and so $\tau(Q_1) \geq 1$ by (1). Consequently, $u_1u_3 \in E$ and so $z_0 \rightarrow (Q, u_4; z_2z_1z_3)$, a contradiction. Hence $z_3u_4 \notin E$ and so $e(z_3, u_1u_3) = 2$. Again, $G_1 \supseteq C_3 \uplus C_4^+$ and so $u_1u_3 \in E$. Thus (20) holds.

To prove the lemma, we shall eliminate each of (15) to (20). We do so by contradiction. First, suppose that (18) or (20) holds. Let $P = u_2z_0z_1z_3$. As $G_1 \not\supseteq 2C_4$, $e(z_1, u_1u_2) = 0$. Thus $e(P, G_1) \leq 15$ and so $e(P, H_1) \geq 8k - 15 = 8(k-2) + 1$. Say w.l.o.g. $e(P, Q_2) \geq 9$. As $[z_2, u_1, u_3, u_4] \cong K_4 > Q_1$ and by (1), $[P, Q_2] \not\supseteq C$ with $C \cong C_3$ and $[V(P \cup Q_2) - V(C)] \geq Q_2$. Then we apply Lemma 3.5 to P and Q_2 and see that either $z_0 \rightarrow (Q_2; z_1z_2z_3)$ or $z_1 \rightarrow (Q_2; z_0u_1u_2)$. Consequently, $G_2 \supseteq 3C_4$, a contradiction.

Next, suppose that either (16) or (17) holds. Let $L = z_0z_1z_3u_2$. As $G_1 \not\supseteq 2C_4$, $e(z_1, u_2u_4) = 0$. Thus $e(L + u_4, G_1) \leq 19$ and so $e(L + u_4, H_1) \geq 10(k-2) + 1$. Say w.l.o.g. $e(L + u_4, Q_2) \geq 11$. Clearly, $[Q_1 - u_2 + z_2] > Q_1$. Then $[L, Q_2] \not\supseteq C$ with $C \cong C_3$ and $[V(L \cup Q_2) - V(C)] \geq Q_2$. If $e(L, Q_2) \geq 9$ then by Lemma 3.5, $\tau(Q_2) = 2$ and there exist two labellings $L = y_1y_2y_3y_4$ and $Q_2 = b_1b_2b_3b_4b_1$ such that one of (a) and (b) in Lemma 3.5 holds w.r.t. L and Q_2. Moreover, if (a) holds then $z_0 \rightarrow (Q_2; z_1z_2z_3)$ or $u_2 \rightarrow (Q_2; z_1z_2z_3)$, and consequently, $G_2 \supseteq 3C_4$, a contradiction. Hence (b) holds. Then $e(z_1, Q_2) \neq 3$ for otherwise $z_1 \rightarrow (Q_2; z_0u_3u_2)$ and so $G_2 \supseteq 3C_4$. Thus $e(L, Q_2) = 9$ with $e(z_3, b_1b_2b_3) = 3$ and $e(z_1, Q_2) = 2$. Thus $e(u_4, Q_2) \geq 11 - 9 = 2$. Then either $z_3 \rightarrow (Q_2; z_1z_2u_4)$ or $u_4 \rightarrow (Q_2; z_1z_2z_3)$, and so $G_2 \supseteq 3C_4$, a contradiction. Hence $e(L, Q_2) \leq 8$ and so $e(u_4, Q_2) \geq 3$. As $x_0 \Rightarrow (Q_1, u_4)$, $u_4 \in \mathcal{T}$. By Lemma 4.1(a), $u_4 \rightarrow Q_2$. As $G_2 \not\supseteq 3C_4$, we see that $u_4 \not\rightarrow (Q_2; P)$ for each $P \in \{z_0z_1z_3, z_0u_3u_2, z_3z_2z_1, z_3u_1u_2, z_1z_3u_2\}$. This means that $u_4 \not\rightarrow (Q_2; vw)$

for each $\{v, w\} \subseteq V(L)$ with $v \neq w$ and $\{v, w\} \neq \{z_0, z_1\}$. Thus $N(b_i, L) = \{z_0, z_1\}$ for each $b_i \in V(Q_2)$ with $e(b_i, L) \geq 2$. As $e(L, Q_2) \geq 11 - e(u_4, Q_2) \geq 7$, it follows that $|I(z_0z_1, Q_2) \cap N(u_4)| \geq 3$. Thus $z_1 \to (Q_2; z_0u_1u_4)$ and so $G_2 \supseteq 3C_4$, a contradiction.

Next, suppose that (19) holds. Let $L_1 = u_4z_3z_1z_0$ and $L_2 = u_3u_2z_0z_1$. As $G_1 \not\supseteq 2C_4$, $e(z_1, u_1u_2u_3) = 0$. Thus $e(L_1, G_1) \leq 15$, $e(L_2, G_1) \leq 15$ and so $e(L_1, H_1) + e(L_2, H_1) \geq 16(k-2)+2$. Say $e(L_1, Q_2)+e(L_2, Q_2) \geq 17$. Clearly, $G_1 - V(L_i) \cong C_4^+$ for $i = 1, 2$. By (1) and Lemma 3.5, $e(L_i, Q_2) \leq 10$ for $i = 1, 2$. Then for some $s \in \{1, 2\}$, $e(L_s, Q_2) = 9 + r$ with $r \in \{0, 1\}$. By Lemma 3.5, $\tau(Q_2) = 2$ and there exist two labellings $L_s = y_1y_2y_3y_4$ and $Q_2 = b_1b_2b_3b_4b_1$ such that one of (a) and (b) in Lemma 3.5 holds w.r.t. L_s and Q_2. First, assume $L_s = L_1$. Then $e(u_2u_3, Q_2) \geq 17 - 9 - r - e(z_0z_1, Q_2) = 8 - r - e(z_0z_1, Q_2)$. As $G_2 \not\supseteq 3C_4$, $[u_2, z_0, z_1, z_3, Q_2] \not\supseteq 2C_4$ and $[u_3, u_4, z_3, z_1, Q_2] \not\supseteq 2C_4$. This implies that $u_2 \not\to (Q_2; z_0z_3)$ and $u_3 \not\to (Q_2; u_4z_1)$. If (b) holds, this further implies that $e(u_2, Q_2) \leq 2$ with $e(u_2, b_3b_4) \leq 1$ and $e(u_3, Q_2) \leq 2$ with $e(u_3, b_3b_4) \leq 1$. Assume $e(u_2, b_1b_2) \neq 0$. Say w.l.o.g. $u_2b_1 \in E$. As $e(b_1, z_1u_4) \geq 1$, we see that $e(z_3, b_2b_3) \leq 1$ for otherwise $z_3 \to (Q_2, b_1; u_2z_0z_1)$ or $z_3 \to (Q_2, b_1; u_2u_3u_4)$ and so $G_2 \supseteq 3C_4$. It follows that $e(L_1, Q_2) = 9$ with $e(z_1, Q_2) = 3$ and $e(z_0, Q_2) = 2$. Thus $z_1 \to (Q_2, b_1; u_2u_1z_0)$ and so $G_2 \supseteq 3C_4$, a contradiction. Hence $e(u_2, b_1b_2) = 0$. Next, assume $e(u_3, b_1b_2) \neq 0$. Say $u_3b_1 \in E$. As $G_2 \not\supseteq 3C_4$, $z_1 \not\to (Q_2, b_1; u_3u_2z_0)$. This implies that $z_0b_1 \notin E$ or $e(z_1, Q_2) \leq 2$. It follows that $e(L_1, Q_2) = 9$ with $e(z_3, b_1b_2b_3) = 3$, $e(u_4, b_1b_2) = 2$ and $e(z_0z_1, Q_2) = 4$. Thus $e(u_2u_3, Q_2) \geq 4$. Hence $e(u_3, Q_2) \geq 3$ and so $u_3 \to (Q_2; u_4z_3z_1)$. Thus $G_2 \supseteq 3C_4$, a contradiction. Therefore $e(u_3, b_1b_2) = 0$ and so $e(u_2u_3, Q_2) \leq 2$. It follows that $e(L_1, Q_2) = 10$, $e(u_2, b_3b_4) = 1$ and $e(u_3, b_3b_4) = 1$. If $u_2b_4 \in E$, then $[u_2, z_0, b_1, b_4] \supseteq C_4$, $[z_1, z_3, b_2, b_3] \supseteq C_4$ and $[z_2, u_1, u_4, u_3] \supseteq C_4$, a contradiction. Hence $u_2b_3 \in E$. Then $z_3 \to (Q_2, b_3; u_2z_0z_1)$ and so $G_2 \supseteq 3C_4$, a contradiction. Hence (a) holds. If $y_1 = z_0$, then $z_0 \to (Q_2; z_1z_2z_3)$ and so $[F, Q_2] \supseteq 2C_4$, a contradiction. Hence $y_1 = u_4$. As $G_2 \not\supseteq 2C_4$, $z_3 \not\to (Q_2; u_2u_3u_4)$ and $z_3 \not\to (Q_2; u_2z_0z_1)$. Thus $i(u_2u_4, Q_2) = 0$ and $i(u_2z_1, Q_2) = 0$. Hence $e(u_2, Q_2) \leq 1$. As $e(u_2u_3, Q_2) \geq 8 - r - e(z_1, Q_2) \geq 4$, $e(u_3, Q_2) \geq 3$. Thus $u_3 \to (Q_2; z_1z_3u_4)$ and so $G_2 \supseteq 3C_4$, a contradiction. Therefore $L_s = L_2$ and $e(z_3u_4, Q_2) \geq 8-r-e(z_0z_1, Q_2)$. If (a) holds, then $y_1 \neq z_1$ for otherwise $z_1 \to (Q_2; z_0u_2u_3)$ by Lemma 3.5 and so $G_2 \supseteq 3C_4$. Thus $y_1 = u_3$ and so $e(z_1, Q_2) = 0$. Consequently, $e(z_3u_4, Q_2) \geq 4$ and if the equality holds then $e(L_2, Q_2) = 10$. As $G_2 \not\supseteq 3C_4$, $u_2 \not\to (Q_2; z_0z_1z_3)$ and so $i(z_0z_3, Q_2) = 0$. Hence $e(z_3, Q_2) \leq 2$ and so $e(u_4, Q_2) \geq 2$. Then $i(u_2u_4, Q_2) \neq 0$. As $G \not\supseteq 3C_4$, $z_0 \not\to (Q_2; u_2u_3u_4)$. This implies $e(z_0, Q_2) \leq 2$. Thus $e(L_2, Q_2) = 9$ and so $e(u_4, Q_2) \geq 3$. Thus $u_4 \to (Q_2; z_0u_2u_3)$ and so $G_2 \supseteq 3C_4$, a contradiction. Hence (b) holds. Then $[z_0, u_2, b_3, b_i] \supseteq C_4$ for each $i \in \{1, 2\}$ and $e(b_i, L_2) = 4$ for some $i \in \{1, 2\}$. Say w.l.o.g. $e(b_1, L_2) = 4$. As $[z_0, u_2, b_3, b_2] \supseteq C_4$ and $G_2 \not\supseteq 3C_4$, we see that $[z_3, z_1, b_1, b_4] \not\supseteq C_4$ and $[u_4, u_3, b_1, b_4] \not\supseteq C_4$. Hence $e(b_4, z_3u_4) = 0$. Suppose that $e(u_2, Q_2) = 3$, ie., $e(u_2, b_1b_2b_3) = 3$. As $G_2 \not\supseteq$

$3C_4$, $u_2 \not\to (Q_2; u_3u_4z_3)$ and $u_2 \not\to (Q_2; z_0z_1z_3)$. As $\{b_1, b_2\} = N(z_0u_3, Q_2)$, it follows that $N(z_3, Q_2) \subseteq \{b_3\}$ and if the equality holds then $z_0b_3 \notin E$ and $e(z_0, b_1b_2) = 2$. Hence $e(u_4, Q_2) \geq 8 - r - e(z_0z_1, Q_2) - e(z_3, Q_2) \geq 2$ and if the last equality holds then $e(L_2, Q_2) = 10$ with $e(z_0, b_1b_2b_3) = 3$ and $e(z_3, Q_2) = 0$. Thus either $e(u_4, Q_2) \geq 3$ and $u_4 \to (Q_2; z_0u_2u_3)$ or $e(z_0, Q_2) \geq 3$ and $z_0 \to (Q_2; u_2u_3u_4)$ and so $G_2 \supseteq 3C_4$, a contradiction. Hence $e(u_2, Q_2) \neq 3$. Thus $e(z_0, Q_2) = 3$ and $e(z_1u_3, b_1b_2) = 4$. As $z_0 \not\to (Q_2; z_1z_3u_4)$ and $z_0 \not\to (Q_2; z_1z_2z_3)$, we see that $e(z_3u_4, b_1b_2) = 0$. With $e(b_4, z_3u_4) = 0$, we obtain $2 \geq e(z_3u_4, Q_2) \geq 8 - e(z_0z_1, Q_2) \geq 3$, a contradiction.

Finally, suppose that (15) holds. Let $L_3 = u_4u_1z_0z_1$ and $L_4 = u_2z_3z_1z_0$. Clearly, $G_1 - V(L_3) \cong G_1 - V(L_4) \cong K_4 > Q_1$. As $G_1 \not\supseteq 2C_4$, $e(z_1, u_2u_4) = 0$. Thus $e(L_3, G_1) \leq 16$ and $e(L_4, G_1) \leq 15$. Then $e(L_3, H_1) + e(L_4, H_1) \geq 16(k-2) + 1$. Say $e(L_3, Q_2) + e(L_4, Q_2) \geq 17$. Let $s \in \{3, 4\}$ be such that $e(L_s, Q_2) \geq 9$. By Lemma 3.5, $\tau(Q_2) = 2$ and there exist two labellings $L_s = y_1y_2y_3y_4$ and $Q_2 = b_1b_2b_3b_4b_1$ such that one of (a) and (b) in Lemma 3.5 holds w.r.t. L_s and Q_2. Thus $e(L_s, Q_2) = 9 + r$ with $r \in \{0, 1\}$. First, assume $L_s = L_4$. Then $e(u_1u_4, Q_2) \geq 8 - r - e(z_0z_1, Q_2)$. If (b) holds then $e(z_1, Q_2) \neq 3$ for otherwise $z_1 \to (Q_2; z_0u_1u_2)$ and so $G_2 \supseteq 3C_4$. Thus $e(L_4, Q_2) = 9$ with $e(z_3, b_1b_2b_3) = 3$ and $e(z_1u_2z_0, b_1b_2) = 6$. Hence $e(u_1u_4, Q_2) \geq 4$. As $G_2 \not\supseteq 3C_4$, $z_3 \not\to (Q_2; u_2u_3u_4)$ and $u_2 \not\to (Q_2; z_3u_3u_4)$. This implies $e(u_4, b_1b_2b_3) = 0$ and so $e(u_4, Q_2) \leq 1$. Thus $e(u_1, Q_2) \geq 3$ and so $G_2 \supseteq 3C_4$ since $u_1 \to (Q_2; z_0z_1z_3)$, a contradiction. Hence (a) holds. As $[F, Q_2] \not\supseteq 2C_4$, $z_0 \not\to (Q_2; z_1z_2z_3)$. Then $y_1 \neq z_0$. Thus $y_1 = u_2$ and so $e(u_1u_4, Q_2) \geq 4$. As $G_2 \not\supseteq 3C_4$, $z_3 \not\to (Q_2; u_2u_3u_4)$ and $z_3 \not\to (Q_2; u_1z_0z_1)$. This implies that $i(u_2u_4, Q_2) = 0$ and $i(u_1z_1, Q_2) = 0$. Hence $e(u_1, Q_2) \leq 2$. Moreover, as $G_2 \not\supseteq 3C_4$, $u_2 \not\to (Q_2; u_4u_3z_3)$ and so $i(u_4z_3, Q_2) = 0$. This implies that $e(u_4, Q_2) \leq 1$ and if equality holds then $e(L_4, Q_2) = 9$ with $e(z_3, Q_2) = 3$. It follows that $e(u_1u_4, Q_2) \leq 3$, a contradiction.

Note that if using Q_i in place of Q_2 in the above argument, then for each Q_i in H_1 with $e(L_4, Q_i) \geq 9$, we see that $e(u_4, Q_i) \leq 1$ and if $e(u_4, Q_i) = 1$ then $e(L_4, Q_i) = 9$.

Next, assume $e(L_3, Q_2) = 9 + r$. Then $e(u_2z_3, Q_2) \geq 8 - r - e(z_0z_1, Q_2)$. First, assume (b) holds w.r.t. L_3 and Q_2. As $G_2 \not\supseteq 3C_4$, $z_0 \not\to (Q_2; z_1z_2u_4)$. Then $e(z_0, Q_2) \neq 3$. Thus $e(L_3, Q_2) = 9$ with $e(u_1, b_1b_2b_3) = 3$ and $e(z_0u_4z_1, b_1b_2) = 6$ by Lemma 3.5. Hence $e(u_2z_3, Q_2) \geq 4$. As $G_2 \not\supseteq 3C_4$, $u_1 \not\to (Q_2; z_0z_1z_3)$ and $z_3 \not\to (Q_2; z_1z_0u_1)$. This implies $e(z_3, Q_2) \leq 1$ and so $e(u_2, Q_2) \geq 3$. Thus $u_2 \to (Q_2; z_1z_0u_1)$ and so $G_2 \supseteq 3C_4$, a contradiction. Hence (a) holds. As $G_2 \not\supseteq 3C_4$, $z_0 \not\to (Q_2; u_1u_3u_4)$ and so $y_1 \neq u_4$. Thus $y_1 = z_1$ and so $e(u_4, Q_2) = 0$. As $z_0 \not\to (Q_2; z_1z_2z_3)$ and $u_1 \not\to (Q_2; z_0z_1z_3)$, we see that $N(z_3, Q_2) \subseteq \{b_4\}$ and if the equality holds then $N(z_1z_0u_1, Q_2) = \{b_1, b_2, b_3\}$. However, if $z_3b_4 \in E$ then $[z_0, z_1, z_3, u_1, Q_2] \supseteq 2C_4$, a contradiction. Hence $e(z_3, Q_2) = 0$. As $G_2 \not\supseteq 3C_4$, $z_0 \not\to (Q_2; z_1z_3u_2)$ and $z_1 \not\to (Q_2; z_0u_1u_2)$. It follows that $N(u_2, Q_2) \subseteq \{b_4\}$. As $G_2 \not\supseteq 3C_4$, $[z_1, z_0, u_1, u_2, Q_2] \not\supseteq 2C_4$ and so $u_2b_4 \notin E$. Thus $e(u_2, Q_2) = 0$. If follows that $r = 1$, i.e., $e(z_0, Q_p) = 4$ and

$e(z_1u_1, b_1b_2b_3) = 6$. Let $R = L_4 + u_4 + b_1$. Clearly, $e(L_4, G_2) \leq 22$, $e(u_4, G_2) = 3$ and $e(b_1, G_2) \leq 8$. Thus $e(R, H_2) \geq 12(k-3) + 3$. Say $e(R, Q_3) \geq 13$. If $e(L_4, Q_3) \geq 9$, then $\tau(Q_3) = 2$ and one of (a) and (b) in Lemma 3.5 holds w.r.t. L_4 and Q_3. As noted above, $e(u_4, Q_3) \leq 1$ and if the equality holds then $e(L_4, Q_3) = 9$. Thus $e(b_1, Q_3) \geq 3$. Consequently, either $b_1 \to (Q_3; z_0z_1z_3)$ or $b_1 \to (Q_3; z_1z_3u_2)$. In the former, $G_3 \supseteq 4C_4$ since $u_1 \to (Q_2, b_1)$ and $z_2 \to (Q_1, u_1)$, and in the latter, $G_3 \supseteq 4C_4$ since $z_0 \to (Q_2, b_1)$ and $z_2 \to (Q_1, u_2)$, a contradiction. Hence $e(L_4, Q_3) \leq 8$ and so $e(u_4b_1, Q_3) \geq 5$. Let $T' = z_0u_1b_1z_0$, $Q_1' = z_1b_2b_4b_3z_1$ and $Q_2' = z_2z_3u_2u_3z_2$. Clearly, $\tau(Q_1') = 1$, $\tau(Q_2') = 2$ and so $(T', Q_1', Q_2', Q_3, \ldots, Q_{k-1})$ is a feasible chain. Thus $u_4 \in \mathcal{T}$. As $z_0 \Rightarrow (Q_2, b_1)$, $b_1 \in \mathcal{T}$. As $e(R, Q_3) \geq 13$, $e(w, R) \geq 4$ for some $w \in V(Q_3)$. Let $\mathcal{S}_1 = \{z_0z_1z_3, z_0u_1u_2, z_0u_1u_4, z_1z_2z_3, z_1z_2u_4, z_1z_3u_2, u_2u_1u_4\}$ and $\mathcal{S}_2 = \{z_0z_1z_3, z_0u_1u_2, z_0b_1z_1, z_0u_1b_1, z_1z_0b_1, z_3u_3u_2, z_3z_1b_1\}$. It is easy to check that $G_2 - V(P + b_1) \supseteq 2C_4$ for each $P \in \mathcal{S}_1$ and $G_2 - V(P + u_4) \supseteq 2C_4$ for each $P \in \mathcal{S}_2$. If $e(b_1, Q_3) \geq 3$ then $b_1 \to Q_3$ by Lemma 4.1(a). As $e(w, R - b_1) \geq 3$, $b_1 \to (Q_3, w; P)$ for some $P \in \mathcal{S}_1$ and so $G_3 \supseteq 4C_4$, a contradiction. Hence $e(b_1, Q_3) \leq 2$ and so $e(u_4, Q_3) \geq 3$. Then $u_4 \to (Q_3, w; P)$ for some $P \in \mathcal{S}_2$ and so $G_3 \supseteq 4C_4$, a contradiction. ∎

Lemma 4.5 *The statement* (14) *does not hold.*

Proof. On the contrary, say (14) holds. W.l.o.g., say $Q = Q_1 = c_1c_2c_3c_4c_1$ with $N(x_0, Q_1) = N(x_3, Q_1) = \{c_1, c_3\}$ and $N(x_2, Q_1) = \{c_1, c_4, c_2\}$. Subject to this condition, we may assume that σ and Q_1 is chosen such that $e(x_1, Q_1)$ is maximal. As $G_1 \not\supseteq 2C_4$, $e(x_1, c_2c_4) = 0$ and so $N(x_1, Q_1) \subseteq \{c_1, c_3\}$. Let $R = V(F) \cup \{c_2, c_4\}$. Clearly, $e(x_0c_2, G_1) + e(R, G_1) \leq 27$ and so $e(x_0c_2, H_1) + e(R, H_1) \geq 16k - 27 = 16(k-2) + 5$. Say $e(x_0c_2, Q_2) + e(R, Q_2) \geq 17$. Clearly, $G_1 - \{x_0, c_1, c_2, c_4\} \supseteq C_4$. As $G_2 \not\supseteq 3C_4$, this implies that $x \not\to (Q_2; yc_1z)$, i.e., $x \not\to (Q_2; yz)$, for each permutation (x, y, z) of $\{x_0, c_2, c_4\}$. We have $\{c_2, c_4\} \subseteq \mathcal{T}$ since $x_0 \Rightarrow (Q_2, c_r)$ for each $r \in \{2, 4\}$. Set $F' = c_4x_2x_1x_3x_2$.

Suppose that $e(u, Q_2) \geq 3$ for some $u \in \{x_0, c_2, c_4\}$. Then $u \to Q_2$ by Lemma 4.1(a). Thus $e(d, T) \leq 1$ for each $d \in V(Q_2)$ and so $e(T, Q_2) \leq 4$. Hence $2e(x_0c_2, Q_2) \geq 17 - e(F', Q_2) \geq 17 - 8 = 9$. This implies that $e(x_0c_2, Q_2) \geq 5$. Assume for the moment that $e(x_0c_2, Q_2) \geq 7$. By Lemma 4.1(a), we see that $\tau(Q_2) = 2$. Since $x_0 \not\to (Q_2; c_2c_4)$ and $c_2 \not\to (Q_2; x_0c_4)$, it follows that $e(c_4, Q_2) = 0$. Thus $e(T, Q_2) \geq 17 - 2e(x_0c_2, Q_2)$. This implies that $N(x_0, Q_2) \cap N(T, Q_2) \neq \emptyset$. For each $x_j \in V(T)$ with $i(x_0x_j, Q_2) \neq 0$, if $j \neq 1$ then $c_2 \to (Q_2; x_0x_1x_j)$ and $x_i \to (Q_1, c_2)$ where $\{i, j\} = \{2, 3\}$, i.e., $G_2 \supseteq 3C_4$, a contradiction. Hence $N(x_0, Q_2) \cap N(x_2x_3, Q_2) = \emptyset$. If $e(x_0c_2, Q_2) = 7$ then $e(T, Q_2) \geq 3$ and so $i(x_0x_1, Q_2) \geq 2$. Consequently, $x_1 \to (Q_2; x_0c_1c_2)$. Thus $G_2 \supseteq 3C_4$ as $[x_2, x_3, c_3, c_4] \supseteq C_4$, a contradiction. Hence $e(x_0c_2, Q_2) = 8$. Then

$e(x_2x_3, Q_2) = 0$. Let $d \in V(Q_2)$ be such that $e(d, x_0x_1) = 2$. Then $[x_0, d, x_1] \cong C_3$, $c_2 \Rightarrow (Q_2, d)$ and $\tau(x_2c_1c_4c_3x_2) = \tau(Q_1) + 1$, contradicting (1). Next, assume that $e(x_0c_2, Q_2) = 6$. Then $e(F', Q_2) \geq 17 - 12 = 5$. As $e(T, Q_2) \leq 4$, $e(c_4, Q_2) \geq 1$. If $e(c_2, Q_2) < 3$ then $e(x_0, Q_2) = 4$ and $e(c_2, Q_2) = 2$. Moreover, $\tau(Q_2) = 2$ by Lemma 4.1(a). Consequently, $x_0 \to (Q_2; c_2c_4)$ or $c_2 \to (Q_2; x_0c_4)$, a contradiction. Hence $e(c_2, Q_2) \geq 3$ and so $c_2 \to Q_2$. As $e(F', Q_2) \geq 5$ there exists $d \in V(Q_2)$ such that $e(d, F') \geq 2$. As $e(d, T) \leq 1$, we have $e(d, c_4x_i) = 2$ for some $x_i \in V(T)$. If $x_i = x_1$ then $c_2 \to (Q_2; x_1x_2c_4)$ and $[x_0, x_3, c_1, c_3] \supseteq C_4$, a contradiction. If $x_i \neq x_1$, let $\{i, j\} = \{2, 3\}$. Then $c_2 \to (Q_2; x_ic_1c_4)$ and $[x_0, x_1, x_j, c_3] \supseteq C_4$, a contradiction. We conclude that $e(x_0c_2, Q_2) = 5$. Thus $e(F', Q_2) \geq 17 - 10 = 7$. As $e(T, Q_2) \leq 4$, $e(c_4, Q_2) \geq 3$. Hence $c_4 \to Q_2$ by Lemma 4.1(a). As $i(x_0c_2, Q_2) \geq 1$, $c_4 \to (Q_2; x_0c_2)$, a contradiction.

Therefore $e(u, Q_2) \leq 2$ for all $u \in \{x_0, c_2, c_4\}$. Then $e(F, Q_2) \geq 17 - e(x_0c_4, Q_2) - 2e(c_2, Q_2) \geq 17 - 8 = 9$ and $e(F', Q_2) \geq 17 - 2e(x_0c_2, Q_2) \geq 9$. If $e(x_0, Q_2) = 0$ then $e(T, Q_2) \geq 9$. Furthermore, applying Lemma 3.2 to F, Q_2 and each $z \in \{c_2, c_4\}$, we would have $e(c_r, Q_2) \leq 1$ for each $r \in \{2, 4\}$ and consequently, $e(x_0c_2, Q_2) + e(R, Q_2) \leq 12 + 2e(c_2, Q_2) + e(c_4, Q_2) \leq 15$, a contradiction. Hence $e(x_0, Q_2) \geq 1$. Similarly, $e(c_4, Q_2) \geq 1$. By Lemma 4.3, there exist two labellings $F = z_0z_1z_2z_3z_1$ and $Q_2 = a_1a_2a_3a_4a_1$ such that one of (3) to (8) holds w.r.t. F and Q_2 where $z_0 = x_0$, $z_1 = x_1$ and $\{z_2, z_3\} = \{x_2, x_3\}$. Since $e(F, Q_2) + e(z_0, Q_2) \geq 17 - 2e(c_2, Q_2) - e(c_4, Q_2) \geq 11$, it follows that $e(z_0, Q_2) = 2$. Since $e(F, Q_2) + e(z_0, Q_2) \leq 12$ and $e(c_4, Q_2) \leq 2$, it follows that $2e(c_2, Q_2) \geq 3$ and so $e(c_2, Q_2) = 2$. We also see that if $e(F, Q_2) = 9$ then $e(c_4, Q_2) = 2$ since $e(c_4, Q_2) \geq 17 - e(F, Q_2) - e(z_0, Q_2) - 2e(c_2, Q_2)$.

As $e(z_0, Q_2) = 2$, each of (3), (7) and (8) does not hold w.r.t. F and Q_2. Thus one of (4) to (6) holds w.r.t. F and Q_2. Then $e(a_1, T) \geq 2$. Hence for each $r \in \{2, 4\}$, $c_r \not\to (Q_2, a_1)$ and so $e(c_r, a_2a_4) \leq 1$. We also note that if (5) holds w.r.t. F and Q_2 then $e(z_i, Q_2) = 3$ for exactly one $z_i \in \{z_2, z_3\}$. This is because (14) holds w.r.t. F and Q_2 by Lemma 4.4. Hence if (5) holds w.r.t. F and Q_2 then there exists exactly one vertex $z_i \in V(T)$ such that $e(z_i, Q_2) = 3$ and we may assume that $e(z_2, a_1a_4a_3) = 3$ and $N(z_3, Q_2) = \{a_1, a_3\}$. Assume for the moment that (6) holds w.r.t. F and Q_2. Then $c_2a_2 \notin E$ for otherwise $[c_2, a_2, z_0, c_1] \supseteq C_4$, $z_1 \to (Q_2, a_2)$ and $[x_2, x_3, c_3, c_4] \supseteq C_4$, i.e., $G_2 \supseteq 3C_4$. Thus $e(c_2, a_1a_4a_3) = 2$ and so $c_2 \to (Q_2, a_2)$. Then $e(a_2, T) \leq 1$. It follows that $e(F, Q_2) = 9$ with $e(a_2, z_1z_2) = 1$, $e(a_1, T) = 3$, $e(a_3, z_1z_2) = 2$ and $z_1a_4 \in E$. If $c_2a_1 \notin E$ then $[c_2, a_3, a_4] \cong C_3$, $[T + a_1] \supseteq K_4 > Q_2$ and $x_0 \Rightarrow (Q_1, c_2)$, contradicting (1). Hence $c_2a_1 \in E$. Then $[c_2, a_1, z_0, c_1] \supseteq C_4$ and $[x_2, x_3, c_3, c_4] \supseteq C_4$. Hence $z_1 \not\to (Q_2, a_1)$ as $G_2 \not\supseteq 3C_4$. This implies $z_1a_2 \notin E$ and so $a_2z_2 \in E$. As $e(F, Q_2) = 9$, $e(c_4, Q_2) = 2$. Since there are exactly two distinct vertices z_i from T with $e(z_i, Q_2) = 3$, it follows, by Lemma 4.3, that (6) holds w.r.t. F' and Q_2. In particular, there exist two labellings $F' = z'_0z'_1z'_2z'_3z'_1$ and $Q_2 = a'_1a'_2a'_3a'_4a'_1$

such that $a'_1a'_3 \in E$, $a'_2a'_4 \notin E$, $e(z'_0, a'_1a'_2) = 2$ and $N(z'_2, Q_2) = \{a'_1, a'_2, a'_3\}$. Clearly, $z'_2 = z_1$, $z'_1 = z_2$ and $\{a'_1, a'_3\} = \{a_1, a_3\}$. As $e(z_1, a_1a_4a_3) = 3$, it follows that $a'_2 = a_4$. Thus $[c_4, a_4, z_1, x_2] \supseteq C_4$, $z_0 \to (Q_2, a_4)$ and $x_3 \to (Q_1, c_4)$, i.e., $G_2 \supseteq 3C_4$, a contradiction. Therefore only one of (4) and (5) holds w.r.t. F and Q_2. When (4) holds w.r.t. F and Q_2, either $e(a_1, F) = 4$ or $e(a_3, F) = 4$. In this case, we may assume that $e(a_1, F) = 4$. We claim that for each $r \in \{2, 4\}$ if $e(c_r, Q_2) = 2$ then $c_ra_1 \in E$ regardless which of (4) and (5) holds w.r.t. F and Q_2. To observe this, we see that if $c_ra_1 \notin E$ then $[c_r, a_2, a_3, a_4] \supseteq C_3$ as $e(c_r, a_2a_4) \le 1$. Moreover, if (4) holds then $a_2a_4 \notin E$ for otherwise $c_r \to (Q_2, a_1; V(T))$. Thus in any case, we have that $[T + a_1] \supseteq K_4 > Q_2$ and $x_0 \Rightarrow (Q_1, c_r)$, contradicting (1). Hence the claim holds. If (4) holds w.r.t. F and Q_2 then $[c_2, a_1, z_0, c_1] \supseteq C_4$ and $[x_2, x_3, c_3, c_4] \supseteq C_4$. Thus $z_1 \not\to (Q_2, a_1)$ as $G_2 \not\supseteq 3C_4$. This implies that $a_2a_4 \notin E$ and $e(z_1, a_2a_4) = 1$. W.l.o.g., say $e(z_1, a_1a_4a_3) = 3$. Then $e(F, Q_2) = 9$ and $e(a_3, F) = 4$. Thus $e(c_4, Q_2) = 2$. If (5) holds w.r.t. F and Q_2 then $e(c_4, Q_2) = 2$ as $e(F, Q_2) = 9$. Thus the above argument implies that if (4) or (5) holds w.r.t. F and Q_2 then $e(c_2c_4, a_1a_3) = 4$ since a_1 and a_3 are in the symmetric position. In any case, let $V(T) = \{x_r, x_s, x_t\}$ be such that $e(x_r, a_1a_4a_3) = 3$ where $x_r \in \{z_1, z_2\}$. Then $N(y, Q_2) = \{a_1, a_3\}$ for all $y \in R - \{x_r\}$. If $x_r = z_1$ then (5) and (14) hold w.r.t F' and Q_2 and if $x_r = z_2$ then (5) and (14) hold w.r.t. F and Q_2. By the assumption on σ and Q_1, we shall have $e(x_1, c_1c_3) = 2$.

Let $S = \{x_0, c_2, c_4, a_2\}$. Then $e(S, G_2) \le 18$ and so $e(S, H_2) \ge 8k - 18 = 8(k - 3) + 6$. Say $e(S, Q_3) \ge 9$. As in the beginning, $x \not\to (Q_3; yc_1z)$, i.e., $x \not\to (Q_3; yz)$, for each permutation (x, y, z) of $\{x_0, c_2, c_4\}$ for otherwise $[G_1, Q_3] \supseteq 3C_4$. As $G_3 \not\supseteq 4C_4$, $x \not\to (Q_3; ya_1z)$, i.e., $x \not\to (Q_3; yz)$, for each $a_2 \in \{x, y, z\} \subseteq S$ with $|\{x, y, z\}| = 3$. We conclude that $x \not\to (Q_3; S - \{x\})$ for all $x \in S$. As $x_0 \Rightarrow (Q_2, a_2)$, we have $a_2 \in \mathcal{T}$. Thus $S \subseteq \mathcal{T}$. As $e(S, Q_3) \ge 9$ and by Lemma 4.1(a), $x \to (Q_3; S - \{x\})$ for each $x \in S$ with $e(x, Q_3) \ge 3$, a contradiction. ∎

Lemma 4.6 *The statement* (13) *does not hold.*

Proof. On the contrary, say (13) holds. W.l.o.g., say $Q = Q_1 = a_1a_2a_3a_4a_1$, $N(x_0, Q_1) = \{a_1\}$, $N(x_2, Q_1) = \{a_1, a_4, a_3\}$, $N(x_3, Q_1) = \{a_1, a_2, a_3\}$ and $a_2a_4 \notin E$. As $G_1 \not\supseteq 2C_4$, $e(x_1, a_2a_4) = 0$. Let $L_1 = x_0x_1x_2a_4$ and $L_2 = x_0x_1x_3a_2$. Then $e(L_1, G_1) \le 15$ and $e(L_2, G_1) \le 15$. Thus $e(L_1, H_1) + e(L_2, H_1) \ge 16(k - 2) + 2$. Say $e(L_1, Q_2) + e(L_2, Q_2) \ge 17$. W.l.o.g., say $e(L_2, Q_2) \ge 9$. Clearly, $G_1 - V(L_2) > Q_1$. By Lemma 3.5, there exist two labellings $L_2 = y_1y_2y_3y_4$ and $Q_2 = b_1b_2b_3b_4b_1$ with $\tau(Q_2) = 2$ such that one of (a) and (b) in Lemma 3.5 holds w.r.t. L_2 and Q_2. We claim that $e(x_0a_4, Q_2) = 0$, $e(x_2x_3, Q_2) = 8$ and $e(x_1a_2, b_1b_2b_3) = 6$. To see this, let $e(L_2, Q_2) = 9 + r$ where $r \in \{0, 1\}$. Then $e(x_2a_4, Q_2) \ge 17 - 9 - r - e(x_0x_1, Q_2) = 8 - r - e(x_0x_1, Q_2)$. Assume that ($b$) holds. Then $e(x_1, Q_2) \neq 3$ for otherwise

$x_1 \to (Q_2; x_0a_1a_2)$ and $[x_2, x_3, a_3, a_4] \supseteq C_4$, i.e., $G_2 \supseteq 3C_4$. Thus $e(x_3, Q_2) = 3$, $e(L_2, Q_2) = 9$ and $e(x_2a_4, Q_2) \geq 4$. As $G_2 \not\supseteq 3C_4$, $x_3 \not\to (Q_2; a_2a_3a_4)$. Thus $i(a_2a_4, Q_2) = 0$, i.e., $e(a_4, b_1b_2) = 0$. If $e(a_4, b_3b_4) = 2$ then $a_4 \to (Q_2, b_1; x_0x_1x_3)$ and so $G_2 \supseteq 3C_4$, a contradiction. Hence $e(a_4, Q_2) \leq 1$ and so $e(x_2, Q_2) \geq 3$. Thus $x_3 \to (Q_2; x_0x_1x_2)$, i.e., $[F, Q_2] \supseteq 2C_4$, a contradiction. Hence (a) holds. Then $y_1 \neq x_0$ for otherwise $x_0 \to (Q_2; x_1x_2x_3)$ and so $[F, Q_2] \supseteq 2C_4$. Thus $y_1 = a_2$ and $e(x_0, Q_2) = 0$. Hence $e(x_2a_4, Q_2) \geq 8 - r - e(x_1, Q_2) \geq 4$ and if the last equality holds then $r = 1$, i.e., $e(L_2, Q_2) = 10$. As $x_3 \not\to (Q_2; a_2a_3a_4)$, $i(a_2a_4, Q_2) = 0$. Thus if $e(a_4, Q_2) \geq 2$ then $a_4 \to (Q_2; a_2a_3x_3)$ and so $G_2 \supseteq 3C_4$, a contradiction. Hence $e(a_4, Q_2) \leq 1$. As $G_2 \not\supseteq 3C_4$, $a_2 \not\to (Q_2; a_4a_3x_2)$. Thus $i(x_2a_4, Q_2) = 0$ and so $e(x_2a_4, Q_2) \leq 4$. It follows that $e(x_2a_4, Q_2) = 4$ and $e(L_2, Q_2) = 10$ (i.e., $e(x_1a_2, b_1b_2b_3) = 6$ and $e(x_3, Q_2) = 4$). As $G_2 \not\supseteq 3C_4$, $a_2 \not\to (Q_2; a_4a_3x_3)$. Thus $e(a_4, Q_2) = 0$ and so $e(x_2, Q_2) = 4$.

Let $R = \{x_0, b_2, b_3, a_2, a_4\}$. Then $e(R, G_2) \leq 29$ and so $e(x_0, H_2) + e(R, H_2) \geq 12k - 31 = 12(k-3) + 5$. Say $e(x_0, Q_3) + e(R, Q_3) \geq 13$. Note that $[x_0, x_1, x_i, a_1] \supseteq C_4$ for $i \in \{2, 3\}$ and $[a_2, a_3, x_3, b_i] \supseteq C_4$ for $i \in \{1, 2, 3\}$. Set $F_1 = x_0x_1b_2b_3x_1$, $Q_2' = x_2x_3b_1b_4x_2$ and $\sigma_1 = (x_0x_1, x_1b_2b_3x_1, Q_1, Q_2', Q_3, \ldots, Q_{k-1})$. Then σ_1 is a strong feasible chain. Let $\mathcal{S}_1 = \{b_2x_1b_3, b_2x_3a_2, b_2x_2a_4, b_3x_3a_2, b_3x_2a_4, a_2a_3a_4\}$, $\mathcal{S}_2 = \{x_0a_1a_2, x_0x_1b_2, x_0x_1b_3, b_2b_4b_3\}$ and $\mathcal{S}_3 = \{x_0a_1a_4, x_0x_1b_2, x_0x_1b_3\}$. Each $P \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ has its two endvertices in R. It is easy to check that $G_2 - V(P + x_0) \supseteq 2C_4$ for each $P \in \mathcal{S}_1$, $G_2 - V(P + a_4) \supseteq 2C_4$ for each $P \in \mathcal{S}_2$ and $G_2 - V(P + a_2) \supseteq 2C_4$ for each $P \in \mathcal{S}_3$. Thus $x_0 \not\to (Q_3; P)$ for each $P \in \mathcal{S}_1$, $a_4 \not\to (Q_3; P)$ for each $P \in \mathcal{S}_2$ and $a_2 \not\to (Q_3; P)$ for each $P \in \mathcal{S}_3$. If $e(x_0, Q_3) \geq 3$ then $x_0 \to Q_3$. As $e(Q_3, R - \{x_0\}) \geq 13 - 2e(x_0, Q_3) \geq 5$, $e(u, R - \{x_0\}) \geq 2$ for some $u \in V(Q_3)$ and so $x_0 \to (Q_3, u; P)$ for some $P \in \mathcal{S}_1$, a contradiction. Hence $e(x_0, Q_3) \leq 2$ and so $e(R, Q_3) \geq 11$. If $e(a_2a_4, Q_3) \leq 4$ then $e(F_1 - x_1, Q_3) \geq 7$. By Lemmas 4.4-4.5, we see that either $e(x_0, Q_3) = 0$ or one of (9) and (13) holds w.r.t. F_1 and Q_3. Thus $e(x_0, Q_3) + e(F_1 - x_1, Q_3) \leq 8$. Consequently, $e(x_0, Q_3) + e(R, Q_3) \leq 8 + e(a_2a_4, Q_3) \leq 12$, a contradiction. Therefore $e(a_2a_4, Q_3) \geq 5$. Let $\{r, t\} = \{2, 4\}$ be such that $e(a_r, Q_3) \geq 3$. Let $\{p, q\} = \{2, 3\}$ be such that $e(x_p, a_1a_ra_3) = 3$ and $e(x_q, a_1a_ta_3) = 3$.

We claim that $a_r \to Q_3$. On the contrary, suppose that $a_r \not\to Q_3$. Then $e(a_r, Q_3) = 3$. Let $Q_3 = u_1u_2u_3u_4u_1$ be such that $e(a_r, u_1u_2u_3) = 3$. Then $u_2u_4 \notin E$. If $a_1a_3 \notin E$, we would have $\tau(x_0a_1x_px_1x_0) \geq \tau(Q_1) = 0$. Then $(a_ra_3, x_qa_ta_3x_q, x_0a_1x_px_1x_0, Q_2, \ldots, Q_{k-1})$ is a strong feasible chain and so $a_r \to Q_3$ by Lemma 4.1(a), a contradiction. Hence $a_1a_3 \in E$. We shall show that $e(u_4, R - \{a_r\}) = 0$. If $e(u_4, F_1 - x_1) \geq 1$, then for some $i \in \{2, 3\}$, say w.l.o.g. $i = 2$, such that $[x_0, x_1, b_2, u_4] \supseteq P_4$. Moreover, $x_p \Rightarrow (Q_2, b_2)$, $\tau(x_qa_1a_ta_3x_q) = \tau(Q_1) + 1$ and $\tau(a_ru_1u_2u_3a_r) = \tau(Q_3) + 1$. This contradicts Lemma 4.2. If $u_4a_t \in E$, then $[x_0x_1, u_4a_t] \supseteq 2P_2$, $\tau(x_2a_1a_3x_3x_2) = \tau(Q_1) + 1$ and

$\tau(a_r u_1 u_2 u_3 a_r) = \tau(Q_3) + 1$, contradicting Lemma 4.2. Therefore $e(u_4, R - \{a_r\}) = 0$. Since $a_4 \not\to (Q_3, u_2; P)$ for each $P \in \mathcal{S}_2$ and $a_2 \not\to (Q_3, u_2; P)$ for each $P \in \mathcal{S}_3$, we see that $u_2 \notin I(x_0 b_i, Q_3)$ for each $i \in \{2, 3\}$. If $I(x_0 b_i, Q_3) \neq \emptyset$ for some $i \in \{2, 3\}$, then $I(x_0 b_i, \{u_1, u_3\}) \neq \emptyset$. W.l.o.g., say $e(u_1, x_0 b_2) = 2$. Then $[a_r, u_2, u_3] \cong C_3$, $[u_1, x_0, x_1, b_2] \supseteq C_4$, $[x_q, a_1, a_t, a_3] \cong K_4$ and $[x_p, b_1, b_3, b_4] \cong K_4$. This violates (2) on σ. Therefore $I(x_0 b_i, \{u_1, u_3\}) = \emptyset$ for each $i \in \{2, 3\}$. We conclude that $i(x_0 b_i, Q_3) = 0$ for each $i \in \{2, 3\}$. It follows that $e(b_2 b_3, Q_3) \leq 2(3 - e(x_0, Q_3))$. This yields that $e(b_2 b_3, Q_3) + 2e(x_0, Q_3) \leq 6$. Consequently, $e(a_r a_t, Q_3) \geq 13 - 6 = 7$. Hence $e(a_t, Q_3) = 4$, a contradiction as $a_t u_4 \notin E$. Therefore $a_r \to Q_3$.

First, assume that $a_r = a_4$. As $a_4 \not\to (Q_3; P)$ for each $P \in \mathcal{S}_2$, we have $i(x_0 y, Q_3) = 0$ for each $y \in \{a_2, b_2, b_3\}$ and $i(b_2 b_3, Q_3) = 0$. Thus $e(x_0, Q_3) + e(b_2 b_3, Q_3) \leq 4$ and $e(x_0, Q_3) + e(a_2, Q_3) \leq 4$. It follows that $e(x_0, Q_3) + e(R, Q_3) \leq 4 + 4 + e(a_4, Q_3) \leq 12$, a contradiction. Therefore we may assume that $a_r = a_2$ and $e(a_4, Q_3) \leq 2$. As $a_2 \not\to (Q_3; P)$ for each $P \in \mathcal{S}_3$, $i(x_0 b_i, Q_3) = 0$ for each $i \in \{2, 3\}$. Then $e(b_2 b_3, Q_3) \leq 2(4 - e(x_0, Q_3))$. Thus $e(b_2 b_3, Q_3) + 2e(x_0, Q_3) \leq 8$. On the other hand, $e(b_2 b_3, Q_3) + 2e(x_0, Q_3) \geq 13 - e(a_2 a_4, Q_3) \geq 13 - 6 = 7$. As $e(a_2 a_4, Q_3) \geq 5$, $i(a_2 a_4, Q_3) \geq 1$. If $e(x_0, Q_3) = 0$ then $e(b_2 b_3, Q_3) \geq 7$ and so $e(b_i, Q_3) = 4$ for some $i \in \{2, 3\}$. Consequently, $b_i \to (Q_3; a_2 a_3 a_4)$, $[x_0, x_1, x_2, a_1] \supseteq C_4$ and $x_3 \to (Q_2, b_i)$, i.e., $G_3 \supseteq 4C_4$, a contradiction. If $e(x_0, Q_3) = 1$, say $d \in V(Q_3)$ with $x_0 d \in E$. Then $e(b_2 b_3, Q_3 - d) \geq 5$. W.l.o.g., say $e(b_2, Q_3 - d) = 3$. If $da_2 \in E$ then $b_2 \to (Q_3; x_0 a_1 a_2)$, $[x_2, x_3, a_3, a_4] \supseteq C_4$ and $x_1 \to (Q_2, b_2)$, i.e., $G_3 \supseteq 4C_4$, a contradiction. Hence $a_2 d \notin E$. Thus $dd^* \in E$ as $a_2 \to Q_3$. Therefore $b_2 \to Q_3$. Thus $b_2 \to (Q_3; a_2 a_3 a_4)$ and it follows, as above, that $G_3 \supseteq 4C_4$, a contradiction. Finally, we have $e(x_0, Q_3) = 2$. Then $e(b_2 b_3, Q_3) \geq 3$. Say $Q_3 = d_1 d_2 d_3 d_4 d_1$ with $x_0 d_1 \in E$. If $x_0 d_3 \in E$ then $e(b_2 b_3, d_2 d_4) \geq 3$ and so $x_0 \to (Q_3; b_2 b_3)$, a contradiction. Therefore $e(x_0, d_2 d_4) = 1$. W.l.o.g., say $x_0 d_2 \in E$. Then $e(b_2 b_3, d_3 d_4) \geq 3$. If $e(a_2, d_1 d_2) = 2$ then $[x_0, d_1, a_2, d_2] \supseteq C_4$, $[b_2, b_3, d_3, d_4] \supseteq C_4$, $x_2 \to (Q_1, a_2)$ and $[x_1, x_3, b_1, b_4] \supseteq C_4$, i.e., $G_3 \supseteq 4C_4$, a contradiction. Hence $e(a_2, d_1 d_2) \leq 1$ and so $e(a_2, Q_3) = 3$. It follows that $e(b_2 b_3, d_3 d_4) = 4$. As $a_2 \to Q_3$, $\tau(Q_3) \geq 1$. Thus $x_0 \to (Q_3; b_2 b_3)$ again, a contradiction. ∎

Lemma 4.7 *In Lemma* 4.3, *none of* (4), (5) *and* (7) *holds.*

Proof. If (5) holds then $e(F - z_1, Q) \geq 7$ with $1 \leq e(z_0, Q) \leq 2$ and none of (9) to (12) holds w.r.t. F and Q. By Lemmas 4.4-4.6, this is impossible. Hence (5) does not hold.

Suppose that (4) holds. W.l.o.g., say $Q = Q_1 = c_1 c_2 c_3 c_4 c_1$ with $c_1 c_3 \in E$, $N(x_i, Q_1) \subseteq \{c_1, c_3\}$ for each $i \in \{0, 2, 3\}$ and $e(F, Q_1) \geq 9$. As $9 \leq e(F, Q_1) \leq 10$, at most one of the ten possible edges between F and Q_1 may miss from G_1. Let

$R = \{x_0, x_2, x_3, c_2, c_4\}$. Clearly, $e(R, G_1) \leq 19$. We claim

$$\text{For each } \{u, v, w\} \subseteq R \text{ with } u \in \{x_0, c_2, c_4\} \text{ and } |\{u, v, w\}| = 3, \\ G_1 - \{u, v, w, z\} \supseteq C_4 \text{ for some } z \in I(vw, G_1 - \{u, v, w\}). \quad (21)$$

To see this, let $u = x_0$ first. If $\{v, w\} = \{x_2, x_3\}$ then obviously, we can take $z = x_1$. If $\{v, w\} = \{c_2, c_4\}$ then take $z = c_1$ since $T + c_3 \supseteq C_4$. Therefore we may assume w.l.o.g. that $v = x_2$ and $w = c_2$ in order to see (21). As $e(x_2x_3, c_1c_3) \geq 3$, $\{x_2c_i, x_3c_j\} \subseteq E$ for some $\{i, j\} = \{1, 3\}$. Say w.l.o.g. $\{x_2c_1, x_3c_3\} \subseteq E$. If $x_1c_4 \in E$ then $[x_3, c_3, c_4, x_1] \supseteq C_4$ and we take $z = c_1$. If $x_1c_4 \notin E$ then $e(x_1, c_1c_2c_3) = 3$ and $e(x_2x_3, c_1c_3) = 4$. Then $[x_3, c_1, c_4, c_3] \supseteq C_4$ and we take $z = x_1$. Next, let $u \in \{c_2, c_4\}$. Say w.l.o.g. $u = c_2$. First, assume $\{v, w\} = \{x_2, x_3\}$. If $e(x_0, c_1c_3) = 2$, take $z = x_1$. If $e(x_0, c_1c_3) \neq 2$ then $e(x_0, c_1c_3) = 1$, $e(x_1, Q_2) = 4$ and $e(x_2x_3, c_1c_3) = 4$. Say w.l.o.g. $x_0c_1 \in E$. Then $[x_0, c_1, c_4, x_1] \supseteq C_4$ and we take $z = c_3$. Next, assume that $v = x_0$ and $w \in \{x_2, x_3\}$. W.l.o.g., say $w = x_2$. If $e(x_3, c_1c_3) = 2$, take $z = x_1$. If $e(x_3, c_1c_3) = 1$ then $e(x_0x_2, c_1c_3) = 4$ and $e(x_1, Q_1) = 4$. Say w.l.o.g. $x_3c_3 \in E$. Then $[x_1, x_3, c_3, c_4] \supseteq C_4$ and we take $z = c_1$. If $\{v, w\} = \{x_0, c_4\}$ then we have either $x_0c_1 \in E$ and $e(c_3, T) = 3$ or $x_0c_3 \in E$ and $e(c_1, T) = 3$. Then we take $z = c_1$ or $z = c_3$ accordingly. Finally, let $\{v, w\} = \{c_4, x_i\}$ for some $i \in \{2, 3\}$. Say w.l.o.g. $\{v, w\} = \{c_4, x_2\}$. We have either $c_1x_2 \in E$ and $e(c_3, x_0x_1x_3) = 3$ or $c_3x_2 \in E$ and $e(c_1, x_0x_1x_3) = 3$. Then we take $z = c_1$ or $z = c_3$ accordingly.

We have $e(x_0, H_1) + e(R, H_1) \geq 12k - 3 - 19 = 12(k-2) + 2$. Say $e(x_0, Q_2) + e(R, Q_2) \geq 13$. As $G_2 \not\supseteq 3C_4$ and by (21), $u \not\to (Q_2; R - \{u\})$ for each $u \in \{x_0, c_2, c_4\}$. If $e(x_0, Q_2) \geq 3$ then $x_0 \to Q_2$ and $e(R - \{x_0\}, Q_2) \geq 13 - 2e(x_0, Q_2) \geq 5$. Thus $x_0 \to (Q_2; R - \{x_0\})$, a contradiction. Hence $e(x_0, Q_2) \leq 2$. Then $e(R, Q_2) \geq 11$.

Suppose $c_2c_4 \notin E$. We claim $\{c_2, c_4\} \subseteq \mathcal{T}$. This is obvious if $e(x_0, Q_1) = 2$ for we have $x_0 \Rightarrow (Q_1, c_i)$ for $i \in \{2, 4\}$ in this situation. If $e(x_0, c_1c_3) = 1$, then $e(x_1, Q_1) = 4$ and $e(x_2x_3, c_1c_3) = 4$. Say w.l.o.g. $x_0c_1 \in E$. Then for each $\{i, j\} = \{2, 4\}$, $\tau(x_0x_1c_ic_1x_0) = \tau(Q_1)$ and so $(c_jc_3, x_2x_3c_3x_2, x_0x_1c_ic_1x_0, Q_2, \ldots, Q_{k-1})$ is a strong feasible chain. Thus $\{c_2, c_4\} \subseteq \mathcal{T}$. If $e(c_i, Q_2) \geq 3$ for some $i \in \{2, 4\}$ then $c_i \to Q_2$ by Lemma 4.1(a). Consequently, $c_i \to (Q_2; R - \{c_i\})$ as $e(R, Q_2) \geq 11$, a contradiction. Hence $e(c_i, Q_2) \leq 2$ for $i \in \{2, 4\}$. Thus $e(F - x_1, Q_2) \geq 13 - e(x_0c_2c_4, Q_2) \geq 7$. As $e(x_0, Q_2) \leq 2$ and by Lemmas 4.4-4.6, either $e(x_0, Q_2) = 0$ or $e(x_0, Q_2) = 1$ with $e(F - x_1, Q_2) = 7$. It follows that $e(x_0, Q_2) + e(R, Q_2) \leq 12$, a contradiction.

Therefore $c_2c_4 \in E$. Clearly, either $x_0c_1 \in E$ and $e(c_3, T) = 3$ or $x_0c_3 \in E$ and $e(c_1, T) = 3$. W.l.o.g., say the former holds. Let $F_1 = x_0c_1c_2c_4c_1$ and $Q_1' = x_1x_2c_3x_3x_1$. Then $\sigma_1 = (x_0c_1, c_1c_2c_4c_1, Q_1', Q_2, \ldots, Q_{k-1})$ is a strong feasible chain. Furthermore, $e(F_1, Q_1') \geq 9$ and (4) holds w.r.t. F_1 and Q_1'. As $e(F - x_1, Q_2) + e(F_1 - c_1, Q_2) = e(x_0, Q_2) + e(R, Q_2) \geq 13$, we may assume w.l.o.g. that $e(F - x_1, Q_2) \geq 7$.

By Lemmas 4.4-4.6, either $e(x_0, Q_2) = 0$ with $e(x_2x_3, Q_2) \geq 7$ or $e(x_0, Q_2) = 1$ with $e(x_2x_3, Q_2 - d) = 6$ and $e(d, x_2x_3) = 0$ for some $d \in V(Q_2)$. Thus $e(c_i, Q_2) \geq 3$ for some $i \in \{2, 4\}$ since $e(x_0, Q_2) + e(R, Q_2) \geq 13$. It follows that $c_i \to (Q_2; x_2x_3)$, a contradiction.

Finally, suppose that (7) holds. Say $Q = Q_1 = a_1a_2a_3a_4a_1$, $N(x_0, Q_1) = \{a_1\}$, $N(x_1, Q_1) = N(x_2, Q_1) = \{a_1, a_2, a_3\}$, $N(x_3, Q_1) = \{a_1, a_3\}$, $a_1a_3 \in E$ and $a_2a_4 \notin E$. Let $F_2 = a_4a_3a_2x_2a_3$ and $Q'_1 = x_0x_1x_3a_1x_0$. Then $\sigma_2 = (a_4a_3, a_3a_2x_2a_3, Q'_1, Q_2, \ldots, Q_{k-1})$ is a strong feasible chain and $a_4 \in \mathcal{T}$. Set $R_1 = \{x_0, a_4, a_2, x_2\}$, $R' = R_1 - \{x_0\}$ and $R'' = R_1 - \{a_4\}$. Then $e(R_1, G_1) = 13$ and so $e(R_1, H_1) \geq 8(k-2)+3$. Say $e(R_1, Q_2) \geq 9$. It is easy to see that $G_1 - V(P + x_0) \supseteq C_4$ for each $P \in \{a_4a_3x_2, a_4a_3a_2, a_2x_1x_2\}$ and $G_1 - V(P + a_4) \supseteq C_4$ for each $P \in \{x_0a_1a_2, x_0x_1x_2, a_2a_3x_2\}$. As $G_2 \not\supseteq 3C_4$, this implies that $x_0 \not\to (Q_2; R')$ and $a_4 \not\to (Q_2; R'')$. As $e(R_1, Q_2) \geq 9$, this implies that $x_0 \not\to Q_2$ and $a_4 \not\to Q_2$. By Lemma 4.1(a), $e(x_0, Q_2) \leq 2$ and $e(a_4, Q_2) \leq 2$. Thus $e(F_2 - a_3, Q_2) = e(R', Q_2) \geq 7$. By Lemmas 4.4-4.6, $e(a_4, Q_2) = 0$ or (9) holds w.r.t. F_2 and Q_2. By Lemma 4.2, $[F_2, Q_2] \not\supseteq P \uplus Q$ such that $P \supseteq 2P_2$, $Q \cong C_4$ and $\tau(Q) = \tau(Q_2) + 2$. Applying Lemma 3.3 to F_2, Q_2 and $z = x_0$, we have a labelling $Q_2 = d_1d_2d_3d_4d_1$ such that $e(a_2x_2, d_2d_3d_4) = 6$ and $x_0d_3 \in E$. Consequently, $a_2 \to (Q_2, d_3; x_0x_1x_2)$ and $x_3 \to (Q_1, a_2)$, i.e., $G_2 \supseteq 3C_4$, a contradiction. ∎

Lemma 4.8 *In Lemma* 4.3, (6) *does not hold.*

Proof. On the contrary, suppose that (6) holds. Say w.l.o.g. $Q = Q_1 = c_1c_2c_3c_4c_1$ such that $e(F, Q_1) \geq 9$, $N(x_0, Q_1) \subseteq \{c_1, c_2\}$, $N(x_2, Q_1) \subseteq \{c_1, c_2, c_3\}$, $N(x_3, Q_1) \subseteq \{c_1\}$, $c_1c_3 \in E$ and $c_2c_4 \notin E$. If $e(x_2, c_2c_3) = 2$ and $e(c_1, x_0x_1x_3) = 3$, let $F' = c_4c_3x_2c_2c_3$ and $Q' = c_1x_0x_1x_3c_1$. Then $(c_4c_3, c_3x_2c_2c_3, Q', Q_2, \ldots, Q_{k-1})$ is a strong feasible chain of G. Moreover, $N(c_4, Q') \subseteq \{c_1, x_1\}$,$N(x_2, Q') \subseteq \{c_1, x_3, x_1\}$, $N(c_2, Q') \subseteq \{c_1, x_0, x_1\}$,$N(c_3, Q') \subseteq \{c_1, x_1\}$ and $e(F', Q') \geq 9$. Thus (5) holds w.r.t. F' and Q', contradicting Lemma 4.7. Therefore either $e(x_2, c_2c_3) = 1$ or $e(c_1, x_0x_1x_3) = 2$. Thus one of (22) to (26) holds:

$$N(x_0, Q) = \{c_1, c_2\}, N(x_1, Q) = \{c_2, c_3, c_4\}, N(x_2, Q) = \{c_1, c_2, c_3\}, x_3c_1 \in E, c_1c_3 \in E; \quad (22)$$

$$N(x_0, Q) = \{c_1, c_2\}, e(x_1, Q) = 4, N(x_2, Q) = \{c_1, c_2, c_3\}, x_3c_1 \notin E, c_1c_3 \in E; \quad (23)$$

$$N(x_0, Q) = \{c_1, c_2\}, e(x_1, Q) = 4, N(x_2, Q) = \{c_1, c_3\}, x_3c_1 \in E, c_1c_3 \in E; \quad (24)$$

$$N(x_0, Q) = \{c_1, c_2\}, e(x_1, Q) = 4, N(x_2, Q) = \{c_1, c_2\}, x_3c_1 \in E, c_1c_3 \in E; \quad (25)$$

$$N(x_0, Q) = \{c_2\}, e(x_1, Q) = 4, N(x_2, Q) = \{c_1, c_2, c_3\}, x_3c_1 \in E, c_1c_3 \in E. \quad (26)$$

If (25) holds, let $F' = x_3x_1c_3c_4x_1$ and $Q'_1 = c_1x_2c_2x_0c_1$. Then (24) holds w.r.t. F' and Q'_1 (by relabelling the vertices accordingly). If (26) holds, let $F'' = x_3x_1c_2x_0x_1$ and $Q''_1 = c_1x_2c_3c_4c_1$. Then (23) holds w.r.t. F'' and Q''_1 (by relabelling the vertices accordingly). Therefore we only need to eliminate each of (22), (23) and (24) in order to prove that (6) does not hold.

Suppose that one of (22), (23) and (24) holds. Let $T_1 = c_1x_0c_2c_1$, $F_1 = T_1 + c_4c_1$, $Q_1' = c_3x_1x_3x_2c_3$, $T_2 = x_1c_2x_0x_1$, $F_2 = T_2 + x_3x_1$ and $Q_1'' = x_2c_1c_4c_3x_2$. Then $\tau(Q_1') = \tau(Q_1'') = 1$. Thus both $\sigma_1 = (c_4c_1, T_1, Q_1', Q_2, \ldots, Q_{k-1})$ and $\sigma_2 = (x_3x_1, T_2, Q_1'', Q_2, \ldots, Q_{k-1})$ are strong feasible chains and so $\{c_4, x_3\} \subseteq \mathcal{T}$. First, assume that (22) or (23) holds. Let $R = \{c_4, x_0, c_2, x_3\}$, $R' = R - \{x_3\}$ and $R'' = R - \{c_4\}$. Then $e(R, G_1) \leq 14$ and so $e(R, H_1) \geq 8(k-2)+2$. Say $e(R, Q_2) \geq 9$. It is easy to see that $G_1 - V(P + x_3) \supseteq C_4$ for each $P \in \{x_0x_1c_4, x_0c_1c_2, c_2c_3c_4\}$ and $G_1 - V(P + c_4) \supseteq C_4$ for each $P \in \{x_0x_1x_3, x_0c_1c_2, c_2x_2x_3\}$. As $G_2 \not\supseteq 3C_4$, this implies that $x_3 \not\to (Q_2; R')$ and $c_4 \not\to (Q_2; R'')$. As $e(R, Q_2) \geq 9$, this further implies that $x_3 \not\to Q_2$ and $c_4 \not\to Q_2$. By Lemma 4.1(a), $e(x_3, Q_2) \leq 2$ and $e(c_4, Q_2) \leq 2$. Since $e(F_1 - c_1, Q_2) = e(R', Q_2) \geq 7$, we see, by Lemmas 4.4-4.6, that either $e(c_4, Q_2) = 0$ or there exists $d \in V(Q_2)$ such that $e(c_4, Q_2) = 1$ $N(x_0, Q_2) = N(c_2, Q_2) = V(Q_2) - \{d\}$. By Lemma 4.2, $[F_1, Q_2] \not\supseteq P \uplus Q$ with $P \cong 2P_2$, $Q \cong C_4$ and $\tau(Q) = \tau(Q_2) + 2$. As $x_3 \not\to (Q_2; x_0c_2)$, we may apply Lemma 3.3 to F_1, Q_2 and $z = x_3$. Thus there exists a labelling $Q_2 = d_1d_2d_3d_4d_1$ such that $e(x_0c_2, d_2d_3d_4) = 6$ and $x_3d_3 \in E$. Then $c_2 \to (Q_2, d_3; x_0x_1x_3)$ and $x_2 \to (Q_2, c_2)$, i.e., $G_2 \supseteq 3C_4$, a contradiction.

Therefore (24) holds. Then $e(F - x_1, G_1) + e(F_1 - c_1, G_1) = 20$ and so $e(F - x_1, H_1) + e(F_1 - c_1, H_1) \geq 12k - 20 \geq 12(k-2) + 4$. Say $e(F - x_1, Q_2) + e(F_1 - c_1, Q_2) \geq 13$. Let $R_1 = \{x_0, x_2, x_3, c_2, c_4\}$. It is easy to check that $G_1 - V(P + x_0) \supseteq C_4$ for each $P \in \{x_3x_1x_2, x_3x_1c_4, x_3x_1c_2, x_2c_3c_4, x_2c_3c_2, c_4c_3c_2\}$. As $G_2 \not\supseteq 3C_4$, this implies that $x_0 \not\to (Q_2; R_1 - \{x_0\})$. As $e(R_1, Q_2) = e(F - x_1, Q_2) + e(F_1 - c_1, Q_2) - e(x_0, Q_2) \geq 9$, this further implies that $x_0 \not\to Q_2$. By Lemma 4.1(a), $e(x_0, Q_2) \leq 2$. Assume for the moment that $e(F - x_1, Q_2) \geq 7$. As $e(x_0, Q_2) \leq 2$, we see, by Lemmas 4.4-4.6, that either $e(x_0, Q_2) = 0$ with $e(x_2x_3, Q_2) \geq 7$ or $e(x_0, Q_2) = 1$ with $e(x_2, Q_2) = e(x_3, Q_2) = 3$. Then $e(x_0, Q_2) + e(F - x_1, Q_2) \leq 8$ and so $e(c_2c_4, Q_2) \geq 13 - 8 = 5$. As $x_3 \in \mathcal{T}$ and $e(x_3, Q_2) \geq 3$, we obtain $x_3 \to (Q_2; c_2c_4)$, i.e., $x_3 \to (Q_2; c_2c_3c_4)$. As $[x_0, x_1, x_2, c_1] \supseteq C_4$, it follows that $G_2 \supseteq 3C_4$, a contradiction. Therefore $e(F - x_1, Q_2) \leq 6$ and so $e(F_1 - c_1, Q_2) \geq 7$. By Lemmas 4.4-4.6, we have that either $e(c_4, Q_2) = 0$ or one of (9) to (12) holds w.r.t. F_1 and Q_2. As $e(x_0, Q_2) \leq 2$, we conclude that $e(x_0, Q_2) \leq 1$ and one of (10) to (12) holds w.r.t. F_1 and Q_2. Thus $e(x_0, Q_2) + e(F_1 - c_1, Q_2) \leq 8$ and so $e(x_2x_3, Q_2) \geq 13 - 8 = 5$. Then $e(x_i, Q_2) \geq 3$ for some $i \in \{2, 3\}$. Thus $x_i \to (Q_2; c_4c_2)$, i.e., $x_i \to (Q_2; c_4c_3c_2)$. Say $\{i, j\} = \{2, 3\}$. Then $[c_1, x_0, x_1, x_j] \supseteq C_4$ and so $G_2 \supseteq 3C_4$, a contradiction. ∎

Lemma 4.9 *Set $G_0 = [F, Q_2]$ and let z_1 and z_2 be two distinct vertices in $G_0 - x_1$ such that if $z_1 \notin V(T)$ then $x_i \to (Q_2, z_1)$ and $e(z_1, T - x_i) \geq 1$ for some $x_i \in V(T)$. In addition, suppose that $G_0 + x \supseteq 2C_4$ for each $x \in V(G) - V(G_0)$ with $e(x, G_0) \geq 2$.*

Then for any $i \in \{1, 3, \ldots, k-1\}$*, there exists no labelling* $Q_i = d_1 d_2 d_3 d_4 d_1$ *such that the following hold:*

(*a*) $x_0 d_1 \in E$, $d_2 d_4 \notin E$, $e(z_1, d_1 d_2 d_3) = 3$, $e(z_2, d_1 d_3) = 2$;

(*b*) *If* $e(x_0, d_1 d_3) = 1$ *then* $Q_i \neq Q_1$, $e(d_2 d_4, Q_1) \leq 4$ *and for some* $y \in V(Q_1)$, $x_0 \Rightarrow (Q_1, y)$, $e(y, d_1 d_3) = 2$.

Proof. Suppose that there exists Q_i as described. Say w.l.o.g. $Q_i = Q_3 = d_1 d_2 d_3 d_4 d_1$. Let $L = [F, Q_2, Q_3]$ if $e(x_0, d_1 d_3) = 2$ and otherwise $L = [F, Q_1, Q_2, Q_3]$. Say $|V(L)| = 4p$. We claim $e(F + d_2 + d_4, L) \leq 12p - 2$. If $e(x_0, Q_3) \geq 3$ then $x_0 \to (Q_3, d_3)$ by Lemma 4.1(a) and so $[F, Q_2, Q_3] \supseteq 3C_4$ since $G_0 + d_3 \supseteq C_4$, a contradiction. Hence $e(x_0, Q_3) \leq 2$. Obviously, $e(F, F) = 8$. As $[F, Q_2] \not\supseteq 2C_4$, $e(F, Q_2) \leq 12$ by Lemma 4.3. As $[F, Q_3] \not\supseteq 2C_4$, if $e(F, Q_3) \geq 9$ then $e(F, Q_3) = 9$, $e(x_0, Q_3) = 1$ and one of (3) and (8) holds w.r.t. F and Q_3 by Lemmas 4.3 and 4.7-4.8. Then by our assumption, $e(y, Q_3) \geq 2$. Thus $[T, Q_3, y] \supseteq 2C_4$ and so $[F, Q_1, Q_3] \supseteq 3C_4$, a contradiction. Hence $e(F, Q_3) \leq 8$. If $e(x_0, d_1 d_3) = 1$ then $e(x_0, Q_1) \geq 2$ as $x_0 \Rightarrow (Q_1, y)$. In this situation, as $[F, Q_1] \not\supseteq 2C_4$, we obtain $e(F, Q_1) \leq 8$ by Lemmas 4.3 and 4.7-4.8. For each $t \in \{2, 4\}$, since $x_0 \Rightarrow (Q_3, d_t)$ or $x_0 \Rightarrow (Q_1, y)$ and $y \Rightarrow (Q_3, d_t)$, we have $d_t \in \mathcal{T}$. Moreover, $G_0 + d_t \not\supseteq 2C_4$ and so $e(d_t, G_0) \leq 1$. If $e(x_0, d_1 d_3) = 2$ then $e(d_2 d_4, L) \leq 6$ and so $e(F, L) + e(d_2 d_4, L) \leq 8 + 12 + 8 + 6 = 12p - 2$ as claimed. Hence assume that $e(x_0, d_1 d_3) = 1$. Then $e(y, d_1 d_3) = 2$. If $e(F, Q_2) + e(d_2 d_4, F \cup Q_2) \leq 14$ then $e(F, L) + e(d_2 d_4, L) \leq 14 + e(F, F \cup Q_1 \cup Q_3) + e(d_2 d_4, Q_1 \cup Q_3) \leq 14 + 24 + 8 = 12p - 2$ as claimed. Therefore we may assume that $e(F, Q_2) + e(d_2 d_4, F \cup Q_2) \geq 15$. As $e(F, Q_2) \leq 12$ and $e(d_i, G_0) \leq 1$ for $i \in \{2, 4\}$, we see that $x_0 d_r \in E$ and $e(d_r, G_0) = 1$ for some $r \in \{2, 4\}$. Let $\{r, t\} = \{2, 4\}$. As $e(x_0, Q_3) \leq 2$, $x_0 d_t \notin E$. It follows that $e(F, Q_2) = 12$ and $e(d_t, G_0) = 1$. Then $e(x_0, Q_2) = 0$ and $e(T, Q_2) = 12$ by Lemmas 4.3. By Lemma 4.1(b), $\tau(Q_2) = 2$. If $e(d_r, G_0 - x_1) = 1$ then $[G_0 + x_0 + d_r] - z_i \supseteq 2C_4$ where $i \in \{1, 2\}$ with $d_r z_i \notin E$. Consequently, $z_i \to (Q_3, d_r)$ and so $[F, Q_2, Q_3] \supseteq 3C_4$, a contradiction. Hence $d_r x_1 \in E$. Thus $d_r = d_4$. Then $[x_0, d_1, d_4, x_1] \supseteq C_4$, $[d_2, d_3, z_1, z_2] \supseteq C_4$ and $G_0 - \{x_1, z_1, z_2\} \supseteq C_4$, a contradiction. Therefore the claim holds. Then $e(F + d_2 + d_4, G - V(L)) = 12k - 12p + 2 = 12(k - p) + 2$. Thus there exists Q_r in $G - V(L)$ such that $e(F + d_2 + d_4, Q_r) \geq 13$.

First, assume that $e(F, Q_r) \leq 8$. Then $e(d_2 d_4, Q_r) \geq 5$. Since $\{d_2, d_4\} \subseteq \mathcal{T}$, we have $d_t \to Q_r$ for some $t \in \{2, 4\}$ by Lemma 4.1(a). Then $T + v \not\supseteq C_4$ and so $e(v, T) \leq 1$ for all $v \in V(Q_r)$ since $x_0 \to (Q_3, d_t)$ or $x_0 \to (Q_1, y)$ and $y \to (Q_3, d_t)$. This yields $e(x_0 d_2 d_4, Q_r) \geq 9$. Then $d_t \to (Q_r; x_0 d_1 d_s)$ where $\{d_s, d_t\} = \{d_2, d_4\}$. As $G_0 + d_3 \supseteq 2C_4$, we obtain $[F, Q_2, Q_3, Q_r] \supseteq 4C_4$, a contradiction.

Therefore $e(F, Q_r) \geq 9$. Then by Lemmas 4.3 and 4.7-4.8, either $e(x_0, Q_r) = 0$ or one of (3) and (8) holds w.r.t. F and Q_r. If (3) or (8) holds w.r.t. F and Q_r, then $e(F, Q_r) = 9$ and $[T, Q_r, d_t] \supseteq 2C_4$ where $d_t \in \{d_2, d_4\}$ with $e(d_t, Q_r) \geq 2$.

Consequently, $[F, Q_3, Q_r] \supseteq 3C_4$ or $[F, Q_1, Q_3, Q_r] \supseteq 4C_4$, a contradiction. Hence $e(x_0, Q_r) = 0$ and so $e(T, Q_r) \geq 9$. Then by Lemma 3.2, $e(d_t, Q_r) \leq 1$ for each $t \in \{2, 4\}$. Thus $e(T, Q_r) \geq 11$ and so $\tau(Q_r) = 2$ by Lemma 4.1(b). By our assumption, there are two distinct vertices x_a and x_b in T such that $z_1 x_a \in E$, $z_1 \notin \{x_a, x_b\}$ and if $z_1 \notin V(T)$ then $x_b \to (Q_2, z_1)$. Let $\{a, b, c\} = \{1, 2, 3\}$. If $e(d_2, Q_r) = 0$ then $e(T, Q_r) = 12$ and $e(d_4, Q_r) = 1$. If $d_2 w \in E$ for some $w \in V(Q_r)$, we claim that $x_a w \notin E$. To see this, assume $x_a w \in E$. Then $[z_1, x_a, w, d_2] \supseteq C_4$. If $z_1 \in V(T)$, then $x_b \to (Q_r, w)$ and so $[T, Q_r, d_2] \supseteq 2C_4$. If $z_1 \notin V(T)$ then $x_b \to (Q_2, z_1)$, $x_c \to (Q_r, w)$ and so $[T, Q_2, Q_r, d_2] \supseteq 3C_4$. It follows that $[F, Q_2, Q_3, Q_r] \supseteq 4C_4$ or $[F, Q_1, Q_2, Q_3, Q_r] \supseteq 5C_4$, a contradiction. Hence $x_a w \notin E$. In any case, we conclude that $e(F + d_2 + d_4, Q_r) = 13$, $e(T, Q_r) \geq 11$ and $e(d_4, Q_r) = 1$. Thus $e(F + d_2 + d_4, G - V(L \cup Q_r)) \geq 12(k - p) + 2 - 13 = 12(k - p - 1) + 1$. Then $e(F + d_2 + d_4, Q_t) \geq 13$ for some Q_t in $G - V(L \cup Q_r)$. By the above argument, we shall have that $e(T, Q_t) \geq 11$, $e(d_4, Q_t) = 1$ and $\tau(Q_t) = 2$. Let $w \in V(Q_r)$ and $v \in V(Q_t)$ be such that $\{v, w\} \subseteq N(d_4)$. As $e(T, Q_r) \geq 11$ and $e(T, Q_t) \geq 11$, there exists $u \in V(T)$ such that $e(u, vw) = 2$. Let $V(T) = \{u, x, z\}$. Then $[u, v, d_4, w] \supseteq C_4$, $x \to (Q_r, w)$ and $z \to (Q_t, v)$, i.e., $[T, Q_r, Q_t, d_4] \supseteq 3C_4$. It follows that $[F, Q_3, Q_r, Q_t] \supseteq 4C_4$ or $[F, Q_1, Q_3, Q_r, Q_t] \supseteq 5C_4$, a contradiction. ∎

In the above proof, the condition that $G_0 + x \supseteq 2C_4$ for all $x \in V(G) - V(G_0)$ with $e(x, G_0) \geq 2$ is used for the estimation of $e(F, Q_2) + e(d_2 d_4, F \cup Q_2)$ and so is the condition of z_2. Moreover, if $e(x_0, d_1 d_3) = 2$ then the condition of z_2 is used only for $G_0 + d_3 \supseteq 2C_4$. Observing this, we have the following corollary.

Corollary 4.9.1 *Set $G_0 = [F, Q_2]$ and let z_1 be a vertex in $G_0 - x_1$ such that if $z_1 \notin V(T)$ then $x_i \to (Q_2, z_1)$ and $e(z_1, T - x_i) \geq 1$ for some $x_i \in V(T)$. Let $i \in \{1, 3, \ldots, k-1\}$. Then the following two statements hold:*

(a) If $G_0 + x \supseteq 2C_4$ for each $x \in V(G) - V(G_0)$ with $e(x, G_0) \geq 2$ then there exists no labelling $Q_i = d_1 d_2 d_3 d_4 d_1$ such that $e(x_0, d_1 d_3) = 2$, $d_2 d_4 \notin E$, $e(z_1, d_1 d_2 d_3) = 3$ and $e(d_3, G_0) \geq 2$.

(b) If there exists a labelling $Q_i = d_1 d_2 d_3 d_4 d_1$ such that $e(x_0, d_1 d_3) = 2$, $d_2 d_4 \notin E$, $e(z_1, d_1 d_2 d_3) = 3$, and $G_0 + d_3 \supseteq 2C_4$, then $e(F + d_2 + d_4, F \cup Q_2 \cup Q_i) \geq 35$.

Proof. The statement (a) is evident. To see (b), suppose that $e(F + d_2 + d_4, F \cup Q_2 \cup Q_i) \leq 34$. Then $e(F + d_2 + d_4, G - V(F \cup Q_2 \cup Q_i)) \geq 12k - 34 = 12(k-3) + 2$. Then a contradiction follows word by word from the last two paragraphs in the proof of Lemma 4.9. ∎

Proof of Claim 2.2. By Lemmas 4.3 and 4.7-4.8, it remains to show that (8) does not hold. On the contrary, say w.l.o.g. $Q = Q_1 = a_1 a_2 a_3 a_4 a_1$, $N(x_0, Q_1) = \{a_1\}$, $e(x_1 x_2, Q_1) = 8$, $a_1 a_3 \in E$ and $e(x_3, Q_1) = 0$. Let $R = \{x_0, x_3, a_2, a_4\}$. Then

$e(R,G_1) \leq 14$ and so $e(R,H_1) \geq 8(k-2)+2$. Say $e(R,Q_2) \geq 9$. Let $F' = x_3x_1a_1x_0x_1$. Clearly, G_1 has an automorphism α such that $\alpha(F) = F'$ and $\alpha(a_i) = a_i (i = 2,3,4)$. Thus $x_3 \in \mathcal{T}$. It is easy to see that for each $\{x,y,z\} \subseteq R$ with $|\{x,y,z\}| = 3$, there exists $v \in \{a_1,a_3,x_1,x_2\}$ such that $e(v,yz) = 2$ and $G_1 - \{x,y,v,z\} \supseteq C_4$. As $G_2 \not\supseteq 3C_4$, this implies that $x \not\rightarrow (Q_2; R-\{x\})$ for each $x \in R$. As $e(R,Q_2) \geq 9$, it follows that $x \not\rightarrow Q_2$ and so $e(x,Q_2) \leq 3$ for all $x \in R$. Furthermore, $e(x_0,Q_2) \leq 2$ and $e(x_3,Q_2) \leq 2$ by Lemma 4.1(a). Thus $e(a_2a_4,Q_2) \geq 5$. W.l.o.g., say $e(a_2,d_1d_2d_3) = 3$ with $Q_2 = d_1d_2d_3d_4d_1$. Then $d_2d_4 \notin E$ as $a_2 \not\rightarrow Q_2$. Moreover, $e(d_2,R-\{a_2\}) \leq 1$ and $e(d_4,R-\{a_2\}) \leq 1$. Thus $e(d_1d_3,R-\{a_2\}) \geq 4$ and so $i(d_1d_3,R-\{a_2\}) \geq 1$. Assume for the moment that $x \notin I(d_1d_3,R-\{a_2\})$ for each $x \in \{x_0,x_3\}$. Then $e(a_4,d_1d_3) = 2$ and $e(d_1d_3,x_0x_3) = 2$. Thus $e(d_2,x_0x_3) = 0$ as $a_4 \not\rightarrow (Q_2; R-\{a_4\})$. As $e(R,Q_2) \geq 9$, $e(x_0x_3,Q_2-d_2) \geq 3$. W.l.o.g., say $e(x_0,Q_2) = 2$. Then $e(x_0,d_td_4) = 2$ for some $t \in \{1,3\}$. Then $e(d_4,x_3a_4) = 0$ as $a_2 \not\rightarrow (Q_2; R-\{a_2\})$. It follows that $e(x_3,d_1d_3) = 1$ and $e(a_4,d_1d_2d_3) = 3$. Thus $[x_0,d_t,d_4] \supseteq C_3$, $\tau(a_2d_2a_4d_3a_2) = \tau(Q_1)$ and $[x_1,a_1,a_3,x_2] \cong K_4 > Q_2$, contradicting (1). Therefore $e(x_i,d_1d_3) = 2$ for some $i \in \{0,3\}$. Say w.l.o.g. $e(x_0,d_1d_3) = 2$. Then for each $i \in \{2,4\}$, $[T,Q_1,d_i] \not\supseteq 2C_4$ and so $e(d_i,Q_1) \leq 1$ and $e(d_i,T) \leq 1$. Thus $e(d_2d_4,G_2) \leq 8$. As $x_0 \not\rightarrow (Q_2; a_2a_4)$, $a_4d_2 \notin E$. As $e(a_4,Q_2) \geq 2$, $e(a_4,d_1d_3) \geq 1$. Say w.l.o.g. $a_4d_3 \in E$. Then $[T,Q_1,d_3] \supseteq 2C_4$. By Corollary 4.9.1($b$), $e(F+d_2+d_4,G_2) \geq 35$. As $[F,Q_2] \not\supseteq 2C_4$ and $e(x_0,Q_2) = 2$, we have $e(F,Q_2) \leq 8$ by Lemmas 4.3 and 4.7-4.8. Thus $e(F,G_2) \leq 25$ and $e(F+d_2+d_4,G_2) \leq 25+8 = 33$, a contradiction. ∎

Lemma 4.10 *Suppose that* $Q_1 = c_1c_2c_3c_4c_1$ *and* $e(x_0x_2,Q_1) \geq 7$ *with* $e(x_0,c_1c_2c_3) = 3$ *and* $x_2c_4 \in E$. *Set* $G_0 = [T,Q_2]$ *and let* $z_1 \in V(G_0) - \{x_1,x_2\}$. *Furthermore, suppose that if* $z_1 \neq x_3$ *then* $x_1z_1 \in E$, $x_3 \rightarrow (Q_2,z_1)$ *and* $G_0 + x \supseteq 2C_4$ *for each* $x \in V(G) - V(G_0)$ *with* $e(x,G_0) \geq 2$. *Then there exists no* Q_r *in* H_1 *such that* $e(x_0z_1,Q_r) = 8$ *and* $e(c_4,Q_r) = 1$.

Proof. On the contrary, suppose that there exists Q_r as described. Let $Q_1' = x_0c_1c_2c_3x_0$ and $F' = c_4x_2x_1x_3x_2$. Then $\sigma' = (c_4x_2, x_2x_1x_3x_2, Q_1', Q_2, \ldots, Q_{k-1})$ is a strong feasible chain. Let $L = [G_1,Q_r]$ if $z_1 = x_3$ and otherwise $L = [G_2,Q_r]$. Let $|V(L)| = 4p$. Say $Q_r = d_1d_2d_3d_4d_1$ with $c_4d_1 \in E$. By Lemma 4.1(a), $c_2c_4 \in E$ and if $e(x_0,Q_1) = 4$ then $\tau(Q_1) = 2$. Thus $x_2 \rightarrow Q_1$. We estimate $e(F'+d_2+d_4,L)$. As $x_0 \rightarrow Q_r$, we see that for each $i \in \{2,4\}$, if $z_1 = x_3$ then $e(d_i,T) = 1$ and if $z_1 \neq x_3$ then $e(d_i,G_0) = 1$ and $e(d_i,T) = 0$. As $x_3 \rightarrow Q_r$ or $x_3 \rightarrow (Q_2,z_1)$ and $z_1 \rightarrow Q_r$, we see that $d_i \not\rightarrow (Q_1; x_0x_1x_2)$ and so $e(d_i,Q_1) \leq 1$ for $i \in \{2,4\}$. Thus $e(d_2d_4,L) \leq 12$. Clearly, $e(F',F') = 8$. As $G_1 \not\supseteq 2C_4$, $e(F',Q_1') \leq 8$. If $z_1 \neq x_3$ then $e(F',Q_2) \leq 12$ as $[F',Q_2] \not\supseteq 2C_4$. It follows that if $z_1 = x_3$ then $e(F'+d_2+d_4,L) \leq 8+8+12+e(F',Q_r) = 28+5 = 12p-3$ and otherwise

$e(F'+d_2+d_4,L) \le 8+8+12+12+e(F',Q_r) = 40+1 = 12p-7$. Thus $e(F'+d_2+d_4, G-V(L)) > 12(k-p)$. Hence $e(F'+d_2+d_4,Q_t) \ge 13$ for some Q_t in $G-V(L)$.

First, assume that $e(F',Q_t) \ge 9$. By Claim 2.2, we see that if $e(c_4,Q_t) \ne 0$ then $e(F',Q_t) = 9$ and $[T,Q_t,d_i] \supseteq 2C_4$ where $d_i \in \{d_2,d_4\}$ with $e(d_i,Q_t) \ge 2$. Consequently, $[F,Q_r,Q_t] \supseteq 3C_4$ as $x_0 \to (Q_r,d_i)$, a contradiction. Hence $e(c_4,Q_t) = 0$ and so $e(T,Q_t) \ge 9$. As $x_0 \Rightarrow Q_r$ and by Lemma 3.2, $e(d_i,Q_t) \le 1$ for $i \in \{2,4\}$. Thus $e(T,Q_t) \ge 11$ and so $\tau(Q_t) = 2$ by Lemma 4.1(b). If $i(d_2d_4,Q_t) = 1$ then $x_2 \to (Q_t; d_2d_3d_4)$ and $[d_1,x_0,x_1,z_1] \supseteq C_4$. If $i(d_2d_4,Q_t) = 0$ then $i(x_1d_j,Q_t) = 1$ for some $j \in \{2,4\}$ and we have that $x_2 \to (Q_t; x_1z_1d_j)$ and $x_0 \to (Q_r,d_j)$. Since $x_3 \to (Q_2,z_1)$ if $z_1 \ne x_3$, we obtain that $[L,Q_t]-V(Q_1) \supseteq pC_4$ in either case, a contradiction. Hence $e(F',Q_t) \le 8$ and so $e(d_2d_4,Q_t) \ge 5$. W.l.o.g., say $e(d_2,Q_t) \ge 3$. As $x_0 \Rightarrow (Q_t,d_2)$, $d_2 \in \mathcal{T}$ and so $d_2 \to Q_t$. Thus $e(u,T) \le 1$ for all $u \in V(Q_t)$. Then $e(d_2d_4c_4,Q_t) \ge 9$. This yields $d_2 \to (Q_t; d_4d_1c_4)$. It follows that $[L,Q_t] \supseteq (p+1)C_4$ since $x_2 \to (Q_1,c_4)$ and $[x_0,x_1,z_1,d_3] \supseteq C_4$ and if $z_1 \ne x_3$ then $x_3 \to (Q_2,z_1)$, a contradiction. ∎

Lemma 4.11 *Let $(u_0u_1, u_1u_2u_3u_1, J_1, \ldots, J_{k-1})$ be a strong feasible chain. Set $G_0 = [V(J_1) \cup \{u_1,u_2,u_3\}]$. Suppose that J_1 has two distinct vertices z_1 and z_2 such that the following three conditions hold:*

(1^0) $\{z_1,z_2\} \subseteq N(u_1)$, $e(u_3,J_1) = 4$, $e(u_1u_2,J_1) \le 6$;

(2^0) $G_0 + x \supseteq 2C_4$ for any $x \in V(G)-V(G_0)$ with $e(x,G_0) \ge 2$;

(3^0) $G_0 - \{z_1,z_2,u_1\} \supseteq C_4$.

Then for each $J_i (i \ge 2)$, there exists no labelling $J_i = d_1d_2d_3d_4d_1$ such that $N(u_0,J_i) = \{d_1,d_4\}$, $d_2d_4 \notin E$, $d_1d_3 \in E$, $N(z_1,J_i) \supseteq \{d_1,d_2,d_3\}$, and $e(z_2,d_2d_3) \ge 1$.

Proof. On the contrary, suppose that there exists J_i as described. Let $F' = u_0u_1u_2u_3u_1$, $P = d_4u_0u_1u_2$ and $L = [F',J_1,J_i]$. We estimate $e(P,L)$. As $u_0 \to (J_i,d_2)$, $G_0 + d_2 \not\supseteq 2C_4$. Thus $e(d_2,G_0-z_1) = 0$ by (2^0) and so $e(d_2,u_1u_2) = 0$. Clearly, $z_1 \xrightarrow{a} (J_i,d_4)$. So by Lemma 4.2, $u_3 \xrightarrow{na} (J_1,z_1)$. This implies that $z_1z_2 \in E$. Thus $[z_1,z_2,d_2,d_3] \supseteq C_4$ and by (3^0), we have $[u_0,u_1,d_1,d_4] \not\supseteq C_4$. Thus $e(u_1,d_1d_4) = 0$. As $u_3 \to (J_1,z_1)$, $z_1 \not\to (J_i,d_4;u_0u_1u_2)$ and so $u_2d_4 \notin E$. By (1^0), it follows that $e(u_1u_2,L) \le 12 + e(u_2,d_1d_3)$. By ($3^0$), $z_1 \not\to (J_i,d_4;u_0u_1z_2)$. Thus $d_4z_2 \notin E$. As $[F',J_i] \not\supseteq 2C_4$, $u_2 \not\to (J_i,d_4;u_0u_1u_3)$. Thus if $e(u_2,d_1d_3) = 2$ then $d_4u_3 \notin E$. It follows that $e(u_2,d_1d_3) + e(d_4,G_0-\{u_1,u_2,z_2\}) \le 5$. Consequently, $e(P,L) \le 12+5+e(u_0,L)+e(d_4,J_i+u_0) = 23$. Then $e(P,G-V(L)) \ge 8k-23 = 8(k-3)+1$ and so $e(P,J_r) \ge 9$ for some J_r in $G-V(L)$. We have that $u_3 \Rightarrow (J_1,z_1)$ and $\tau(z_1d_1d_2d_3z_1) = \tau(J_i)+1$. By (1), $[P,J_r] \not\supseteq C \uplus Q$ such that $C \cong C_3$ and $Q > J_r$. By Lemma 3.5, either $u_1 \to (J_r;u_0d_4)$ or $u_0 \to (J_r;u_1u_2)$. In the former, $[u_1,u_0,d_1,d_4,J_r] \supseteq 2C_4$. Consequently, $[L,J_r] \supseteq 4C_4$ since $[z_1,z_2,d_2,d_3] \supseteq C_4$ and

$G_0 - \{z_1, z_2, u_1\} \supseteq C_4$, a contradiction. In the latter, $[F', J_r] \supseteq 2C_4$, a contradiction. ∎

Proof of Claim 2.3 and Claim 2.4. We prove them by contradiction. Say $Q = Q_1 = c_1c_2c_3c_4c_1$. To prove Claim 2.3, we assume that $e(x_0, Q_1) = 4$, $x_1c_2 \in E$ and $e(x_2, Q_1) \geq 2$. By Lemma 4.1(a), $\tau(Q_1) = 2$. Thus we may assume $e(x_2, c_3c_4) = 2$. To prove Claim 2.4, we may assume $e(x_0x_2, Q_1) \geq 7$. Moreover, if $e(x_0, Q_1) = 4$ then $e(x_2, c_2c_3c_4) = 3$ and if $e(x_0, Q_1) = 3$ then $e(x_0, c_1c_2c_3) = 3$. In any case, if $e(x_0, Q_1) = 4$ then $\tau(Q_1) = 2$ and $V(Q_1) \subseteq \mathcal{T}$ and if $e(x_0, Q_1) = 3$ then $c_2c_4 \in E$ and $c_4 \in \mathcal{T}$. Note that $x_2 \to Q_1$ if $e(x_0x_2, Q_1) \geq 7$. As $G_1 \not\supseteq 2C_4$, $e(x_3, Q_1) = 0$ and $i(x_1x_2, Q_1) = 0$. Let $T' = x_2x_3x_1x_2$, $F' = T' + c_4x_2$ and $Q_1' = x_0c_1c_2c_3x_0$. Then $\tau(Q_1') = \tau(Q_1)$ and so $\sigma' = (c_4x_2, T', Q_1', Q_2, \ldots, Q_{k-1})$ is a strong feasible chain. It is easy to check that $e(F' - x_2, G_1) + e(F - x_1, G_1) \leq 23$ and so $e(F' - x_2, H_1) + e(F - x_1, H_1) \geq 12(k-2) + 1$. Say w.l.o.g. $r_0 = e(F' - x_2, Q_2) + e(F - x_1, Q_2) \geq 13$.

Subclaim (a). It holds that $e(x_0c_4, Q_2) = 0$ and so $e(x_3, Q_2) + e(T, Q_2) \geq 13$.

Proof. On the contrary, suppose that $e(x_0c_4, Q_2) \geq 1$. Assume that $e(x_3, Q_2) \leq 2$. Then $e(F + c_4, Q_2) \geq 13 - e(x_3, Q_2) \geq 11$. Suppose that $e(v, Q_2) \geq 3$ for some $v \in \{x_0, c_4\}$. Then $v \to Q_2$ by Lemma 4.1(a). Thus $e(d, T) \leq 1$ for all $d \in V(Q_2)$. Then $e(x_0c_4, Q_2) \geq 7$. By Lemma 4.1(a), $\tau(Q_2) = 2$. Clearly, $e(x_1x_2, Q_2) \leq 4 - e(x_3, Q_2)$. Then $e(x_0x_3, Q_2) \geq 13 - e(x_3, Q_2) - (4 - e(x_3, Q_2)) - e(c_4, Q_2) \geq 9 - e(c_4, Q_2) \geq 5$. Hence $i(x_0x_3, Q_2) \geq 1$ and so $c_4 \to (Q_2; x_0x_1x_3)$. Then $x_2 \not\to (Q_1, c_4)$ for otherwise $G_2 \supseteq 3C_4$. By our assumption on Q_1, we shall have that $e(x_0, Q_1) = 4$, $x_1c_2 \in E$ and $e(x_2, c_3c_4) = 2$. Then $e(x_3, Q_2) < 2$ for otherwise $x_3 \to (Q_2; x_0c_1c_4)$ and $[x_1, x_2, c_2, c_3] \supseteq C_4$, i.e., $G_2 \supseteq 3C_4$. As $r_0 \geq 13$, it follows that $e(x_3, Q_2) = 1$, $e(x_0c_4, Q_2) = 8$ and $e(x_1x_2, Q_2) = 3$. If $e(x_2, Q_2) \geq 2$ then $x_2 \to (Q_2; x_0x_1x_3)$ and if $e(x_1, Q_2) \geq 2$ then $x_1 \to (Q_2; c_4x_2x_3)$, i.e., $[F, Q_2] \supseteq 2C_4$ or $[F', Q_2] \supseteq 2C_4$, a contradiction. Therefore $e(v, Q_2) \leq 2$ for $v \in \{x_0, c_4\}$. Then $e(F, Q_2) \geq 9$. By Claim 2.2, we see that $e(x_0, Q_2) = 0$ for otherwise $e(c_4, Q_2) = 2$ and $[T, Q_2, c_4] \supseteq 2C_4$. Hence $e(F', Q_2) \geq 11$. As $e(x_0c_4, Q_2) \geq 1$, $[F', Q_2] \supseteq 2C_4$ by Claim 2.2, a contradiction.

Therefore $e(x_3, Q_2) \geq 3$. If $e(F, Q_2) \geq 9$, then by Claim 2.2, $e(x_0, Q_2) = 0$ for otherwise $e(x_3, Q_2) = 2$. Thus $e(c_4, Q_2) \geq 1$ as $e(x_0c_4, Q_2) \geq 1$. Then $e(F', Q_2) \geq 10$ and so $[F', Q_2] \supseteq 2C_4$ by Claim 2.2, a contradiction. Therefore $e(F, Q_2) \leq 8$. Similarly, $e(F', Q_2) \leq 8$. It follows that $e(x_0x_3, Q_2) \geq 13 - e(F', Q_2) \geq 5$ and $e(x_3c_4, Q_2) \geq 13 - e(F, Q_2) \geq 5$. In particular, we obtain $i(x_0x_3, Q_2) \geq 1$ and $i(x_3c_4, Q_2) \geq 1$. As $r_0 \geq 13$, $e(F' - x_2, Q_2) \geq 7$ or $e(F - x_1, Q_2) \geq 7$. First, assume that $e(F' - x_2, Q_2) \geq 7$. Then by Lemmas 4.4-4.6, one of (9) to (12) holds w.r.t. F' and Q_2. As $e(x_3c_4, Q_2) \geq 5$, (9) does not hold w.r.t. F' and Q_2. Thus $e(v, Q_2) \geq 3$ and $v \to Q_2$ for each $v \in \{c_4, x_3\}$. Since $i(x_0x_3, Q_2) \geq 1$, $c_4 \to (Q_2; x_0x_1x_3)$. As $G_2 \not\supseteq 3C_4$, $x_2 \not\to (Q_1, c_4)$. By our assumption on Q_1, we have that $e(x_0, Q_1) = 4$, $x_1c_2 \in E$ and

$e(x_2, c_3c_4) = 2$. As $e(x_0c_4, Q_2) \geq 13 - e(x_3, Q_2) - e(T, Q_2) \geq 13 - 4 - 4 = 5$, we have that $i(x_0c_4, Q_2) \geq 1$. Then $x_3 \to (Q_2; x_0c_1c_4)$ and $[x_1, x_2, c_3, c_2] \supseteq C_4$, i.e., $G_2 \supseteq 3C_4$, a contradiction. Hence $e(F - x_1, Q_2) \geq 7$. By Lemmas 4.4-4.6, one of (9) to (12) holds w.r.t. to F and Q_2. As $e(x_0x_3, Q_2) \geq 5$, (9) does not hold w.r.t. F and Q_2. Thus $e(v, Q_2) \geq 3$ and $v \to Q_2$ for each $v \in \{x_0, x_3\}$. Again, $e(x_0c_4, Q_2) \geq 13 - e(x_3, Q_2) - e(T, Q_2) \geq 5$. Then $x_1c_2 \notin E$ for otherwise $G_2 \supseteq 3C_4$ as above. Thus $e(x_2, Q_1) \geq 3$ and so $x_2 \to (Q_1, c_4)$. Then $e(c_4, Q_2) < 2$ for otherwise $c_4 \to (Q_2; x_0x_1x_3)$ and so $G_2 \supseteq 3C_4$. As $r_0 \geq 13$, it follows that $e(x_3, Q_2) = 4$, $e(c_4, Q_2) = 1$ and $e(x_0, Q_2) = 4$. This contradicts Lemma 4.10 with $z_1 = x_3$. $\square$

Subclaim (*b*). Suppose that Claim 2.3 holds. Then there exists Q_p in H_1 such that either $e(x_0c_4, Q_p) = 0$, $e(x_1x_3, Q_p) = 7 + q$ and $e(T, Q_p) \geq 10 - q$ for some $q \in \{0, 1\}$, or one of the following statements holds:

(1^0) $e(c_4x_1, Q_p) = 7 + t$ and $e(x_0, Q_p) \geq 3 - 2t$, $e(x_2x_3, Q_p) = 0$;

(2^0) $e(c_4, Q_p) = 4$, $e(x_0, Q_p) \geq 3$, $e(x_1, Q_p) \geq 3$, and $e(x_2x_3, Q_p) = 0$;

(3^0) $e(x_0, Q_p) = 4$, $e(c_4, Q_p) = 3$, $e(x_1, Q_p) \geq 3$ with $e(x_1x_2, Q_p) = 4$, and $e(x_3, Q_p) = 0$.

Proof. By the assumed Claim 2.3, $e(x_1, Q_1) = 0$. Then $e(F' - x_2, G_1) + e(F + c_4, G_1) \leq 31$ and so $e(F' - x_2, H_1) + e(F + c_4, H_1) \geq 16(k - 2) + 1$. Thus there exists Q_p in H_1 such that $r_1 = e(F' - x_2, Q_p) + e(F + c_4, Q_p) \geq 17$. Let $G' = [G_1, Q_p]$. If $e(c_4x_0, Q_p) = 0$ then $r_1 = 2e(x_1x_3, Q_p) + e(x_2, Q_p) \geq 17$. Thus $e(x_1x_3, Q_p) = 7 + q$ and $e(T, Q_p) \geq 17 - 7 - q = 10 - q$ for some $q \in \{0, 1\}$ and so the lemma holds. We now assume $e(x_0c_4, Q_p) \geq 1$. First, suppose $e(F' - x_2, Q_p) \geq 7$. By Lemmas 4.4-4.6, there exist two labellings $F' = z_0z_1z_2z_3z_1$ and $Q_p = u_1u_2u_3u_4u_1$ such that either $e(c_4, Q_p) = 0$ or one of (9) to (12) holds w.r.t. F' and Q_p. Then $e(x_i, Q_p) \neq 2$ for $i \in \{1, 3\}$. If $e(c_4, Q_p) \leq 1$ then $e(x_1x_3, Q_p) + 2e(c_4, Q_p) \leq 8$ and so $e(F, Q_p) \geq 17 - 8 = 9$. By Claim 2.2, $e(x_0, Q_p) = 0$ for otherwise $e(x_3, Q_p) = 2$. Thus $e(T, Q_p) \geq 9$ and $e(c_4, Q_p) \geq 1$. By Claim 2.2, $[F', Q_p] \supseteq 2C_4$, a contradiction. Hence $e(c_4, Q_p) \geq 3$ and so $c_4 \to Q_p$. Thus $e(u_i, T) \leq 1$ for all $u_i \in V(Q_p)$ and so $e(F', Q_p) \leq 8$. If $z_2 = x_3$ then $e(x_3, Q_p) \geq 3$. As $G' \not\supseteq 3C_4$, $c_4 \not\to (Q_p; x_0x_1x_3)$ and so $i(x_0x_3, Q_p) = 0$. Thus $e(x_0, Q_p) + e(F' - x_2) \leq 8$ and so $e(F', Q_p) \geq 17 - 8 = 9$, a contradiction. Hence $z_2 = x_1$. If $e(x_2, Q_p) = 1$ then $e(x_1, Q_p) = 3$ and $e(c_4, Q_p) = 4$, contradicting the assumed Claim 2.3 (w.r.t. F' and Q_p). Hence $e(x_2, Q_p) = 0$. If $e(x_3, Q_p) = 1$ then (12) holds w.r.t. F' and Q_2 such that $x_3u_4 \in E$ and $e(c_4x_1, u_1u_2u_3) = 6$. Thus $e(x_0, u_1u_2u_3) \geq 17 - 2e(F', Q_p) = 3$. Then $[x_3, u_4, u_3, x_1] \supseteq C_4$, $[x_0, u_1, c_4, u_2] \supseteq C_4$ and $x_2 \to (Q_1, c_4)$, i.e., $G' \supseteq 3C_4$, a contradiction. Hence $e(x_3, Q_p) = 0$. Say $e(c_4x_1, Q_p) = 7 + t$ with $t \in \{0, 1\}$. Then $e(x_0, Q_p) \geq 17 - 2(7 + t) = 3 - 2t$, i.e., ($1^0$) holds.

Next, suppose $e(F' - x_2, Q_p) \leq 6$. Then $e(F + c_4, Q_p) \geq 11$. If $e(F, Q_p) \geq 9$ then by Claim 2.2, $e(x_0, Q_p) = 0$ for otherwise $e(c_4, Q_p) \geq 2$ and so $[T, Q_p, c_4] \supseteq 2C_4$. But then $e(F', Q_p) \geq 11$ and so $e(c_4, Q_p) = 0$ by Claim 2.2. Thus $e(c_4x_0, Q_p) = 0$, a contradiction. Hence $e(F, Q_p) \leq 8$ and so $e(c_4, Q_p) \geq 3$. By Lemma 4.1(a), $c_4 \to Q_p$. Then $e(v, T) \leq 1$ for all $v \in V(Q_p)$. Hence $e(x_0c_4, Q_p) \geq 7$ and $4 \geq e(T, Q_p) \geq 3$. As $e(w, Q_p) = 4$ for some $w \in \{c_4, x_0\}$, $\tau(Q_p) = 2$ by Lemma 4.1(a). As $G' \not\supseteq 3C_4$, $c_4 \not\to (Q_p; x_0x_1x_3)$ and so $i(x_0x_3, Q_p) = 0$. If $e(x_3, Q_p) \geq 1$ then $e(x_0, Q_p) = 3$, $e(x_3, Q_p) = 1$, $e(c_4, Q_p) = 4$ and $e(x_1x_2, Q_p) = 3$. Then $e(x_1, Q_p) = 0$ for otherwise $[x_1, x_3, u, v] \supseteq C_4$ for an edge uv of $[Q_p]$ and so $[x_1, x_3, x_0, c_4, Q_p] \supseteq 2C_4$, a contradiction. Thus $e(x_2, Q_p) = 3$, and consequently, $r_1 = 16$, a contradiction. Hence $e(x_3, Q_p) = 0$. As $r_1 \geq 17$, $2e(x_1, Q_p) + e(x_2, Q_p) \geq 17 - e(x_0, Q_p) - 2e(c_4, Q_p)$. This implies that $e(x_1, Q_p) \geq 1$. First, assume $e(x_0c_4, Q_p) = 8$. By the assumed Claim 2.3, $e(x_2, Q_p) \leq 1$. If $e(x_2, Q_p) = 1$, we apply the assumed Claim 2.3 to F' and Q_p and see that $e(x_1, Q_p) = 1$. Thus $r_1 = 15$, a contradiction. Hence $e(x_2, Q_p) = 0$. Then $2e(x_1, Q_p) \geq 5$. Thus $e(x_1, Q_p) \geq 3$ and so (2^0) holds. Next, assume $e(x_0, Q_p) = 3$ and $e(c_4, Q_3) = 4$. Then $2e(x_1, Q_p) + e(x_2, Q_p) \geq 6$. As $i(x_1x_2, Q_p) = 0$, $e(x_1x_2, Q_p) \leq 4$ and so $e(x_1, Q_2) \geq 2$. Applying the assumed Claim 2.3 to F' and Q_p, we obtain $e(x_2, Q_p) = 0$. Thus $e(x_1, Q_p) \geq 3$ and so (2^0) holds. Finally, assume $e(x_0, Q_p) = 4$ and $e(c_4, Q_p) = 3$. Then $2e(x_1, Q_p) + e(x_2, Q_p) \geq 7$. Thus $e(x_1, Q_p) \geq 3$. In addition, if $e(x_1, Q_p) = 3$ then $e(x_2, Q_p) = 1$. Thus (3^0) holds. This proves Subclaim (b). □

By Subclaim (a), $e(x_3, Q_2) + e(T, Q_2) \geq 13$. This yields that $e(x_1x_3, Q_2) \geq 7$ or $e(x_2x_3, Q_2) \geq 7$. Accordingly, we divide our proof into two cases. Case I will be readily reduced to Case II by choosing an appropriate strong feasible chain.

Case I. $e(x_2x_3, Q_2) \geq 7$.

To reduce this case to Case II, we assume that we will arrive a contradiction in Case II. Thus $e(x_1x_3, Q_2) \leq 6$. If $e(x_1, Q_1) \geq 1$, then by the assumption on Q_1, $e(x_0, Q_1) = 4$, $e(x_2, c_3c_4) = 2$, $x_1c_2 \in E$. Then $e(x_1, Q_1') \geq 2$, $e(x_2, Q_1') \geq 1$, $e(c_4, Q_1') = 4$. With F, Q_1 and σ replaced by F', Q_1' and σ', this goes to Case II (if necessary, exchanging the subscripts of x_1 and x_2). Suppose that $e(x_1, Q_1) = 0$. Then $e(x_0x_2, Q_1) \geq 7$. If there exists Q_p in H_1 such that $e(x_1x_3, Q_p) = 7 + q$ and $e(T, Q_p) \geq 10 - q$ for some $q \in \{0, 1\}$, then $e(x_3, Q_p) + e(T, Q_p) \geq 17 - e(x_1, Q_p) \geq 13$. Thus we may replace Q_2 by Q_p and go to Case II. If there exists no such Q_p in H_1, then by Subclaim (b), there exists Q_p in H_1 such that Q_p satisfies one of (1^0)-(3^0). If $e(c_4x_1, Q_p) \geq 7$, then replacing F, Q_1 and Q_p by F', Q_p and Q_1', we go to Case II. If $e(c_4x_1, Q_p) \leq 6$, then (3^0) holds with $e(x_0, Q_p) = 4$, $e(c_4, Q_p) = 3$, $e(x_1, Q_p) = 3$ and $e(x_2, Q_p) = 1$. By Lemma 4.1(a), $\tau(Q_p) = 2$. Let $c \in N(x_2, Q_p)$ and $F'' = T + x_2c$. Let Q_p' be a 4-cycle in $[Q_p - c + x_0]$. Then $\tau(Q_p') = 2$, $e(c, Q_p') = 4$ and $e(x_1, Q_p') = 4$.

Replacing F, Q_1 and Q_p by F'', Q'_p and Q_1, we go to Case II.

Case II. $e(x_1x_3, Q_2) \geq 7$.

We may assume that $e(T, Q_2) \geq e(T, Q_i)$ for all Q_i in H_1 with $e(x_1x_3, Q_i) \geq 7$ and $e(x_3, Q_i)+e(T, Q_i) \geq 13$. Clearly, if $e(x_1x_3, Q_2) = 8$ then $e(x_2, Q_2) \geq 1$. If $e(x_3, Q_2) = 3$ then $e(x_2, Q_2) \geq 3$. If $e(x_1, Q_2) = 3$ then $e(x_2, Q_2) \geq 2$. Let $G_0 = [T, Q_2]$. By (1), $G_0 \not\supseteq C \cong C_3$ such that $G_0 - V(C) > Q_2$. If $\{i, j\} = \{1, 3\}$ with $e(x_i, Q_2) = 4$ and $u \in I(x_2x_j, Q_2)$, then $[x_2, x_j, u] \cong C_3$ and so $x_i \overset{na}{\to} (Q_2, u)$. This implies that $uu^* \in E$. Hence $\tau(Q_2) \geq 1$. We claim there exists a labelling $V(Q_2) = \{a_1, a_2, a_3, a_4\}$ such that $a_1a_2a_3a_4a_1$ is a 4-cycle in $[Q_2]$ and one of (27) to (34) holds:

$$e(x_1x_3, Q_2) = 8, x_2a_1 \in E, N(x_2, Q_2) \subseteq \{a_1, a_3\}, a_1a_3 \in E, a_2a_4 \notin E; \tag{27}$$
$$e(x_1x_3, Q_2) = 8, x_2a_1 \in E, \tau(Q_2) = 2; \tag{28}$$
$$e(x_3, Q_2) = 4, N(x_1, Q_2) = \{a_2, a_3, a_4\}, N(x_2, Q_2) = \{a_1, a_3\}, a_1a_3 \in E, a_2a_4 \notin E; \tag{29}$$
$$e(x_3, Q_2) = 4, N(x_1, Q_2) = \{a_1, a_4, a_3\}, N(x_2, Q_2) = \{a_1, a_3\}, a_1a_3 \in E, a_2a_4 \notin E; \tag{30}$$
$$e(x_3, Q_2) = 4, N(x_1, Q_2) = \{a_1, a_4, a_3\}, \{a_1, a_4\} \subseteq N(x_2, Q_2), \tau(Q_2) = 2; \tag{31}$$
$$e(x_3, Q_2) = 4, N(x_1, Q_2) = \{a_2, a_3, a_4\}, \{a_1, a_4\} \subseteq N(x_2, Q_2), \tau(Q_2) = 2; \tag{32}$$
$$e(x_1, Q_2) = 4, N(x_3, Q_2) = N(x_2, Q_2) = \{a_1, a_2, a_3\}, \tau(Q_2) = 2; \tag{33}$$
$$e(x_1, Q_2) = 4, N(x_3, Q_2) = \{a_1, a_2, a_3\}, \{a_1, a_2, a_4\} \subseteq N(x_2, Q_2), \tau(Q_2) = 2. \tag{34}$$

To observe this, we see that (27) holds if $e(x_1x_3, Q_2) = 8$ with $\tau(Q_2) = 1$ and (28) holds if $e(x_1x_3, Q_2) = 8$ with $\tau(Q_2) = 2$. If $e(x_3, Q_2) = 4$, $e(x_1, Q_2) = 3$ and $\tau(Q_2) = 1$ then (29) or (30) holds. If $e(x_3, Q_2) = 4$, $e(x_1, Q_2) = 3$ and $\tau(Q_2) = 2$ then (31) holds if $N(x_2, Q_2) \subseteq N(x_1, Q_2)$ and otherwise (32) holds. If $e(x_3, Q_2) = 3$ and $e(x_1, Q_2) = 4$ then $e(x_2, Q_2) \geq 3$ and $\tau(Q_2) = 2$ by Lemma 4.1(b). In this last situation, we see that (33) holds if $N(x_2, Q_2) = N(x_3, Q_2)$ and otherwise (34) holds. Clearly, $x_3 \to Q_2$ in any case. We now choose two vertices z_1 and z_2 from Q_2 such that $\{z_1, z_2\} = \{a_3, a_4\}$ if $e(T, Q_2) \leq 10$. If $e(T, Q_2) \geq 11$, then $\tau(Q_2) = 2$ and we let $\{z_1, z_2\} \subseteq N(x_1, Q_2)$ such that $G_0 - \{x_1, z_1, z_2\} \cong K_4$. We claim

$$G_0 - \{z_1, z_2, x_1\} \supseteq C_4^+, G_0 - \{z_i, x_1, x_2\} \supseteq C_4 (i = 1, 2), G_0 - \{z_1, z_2, x_2\} \supseteq C_4; \tag{35}$$
$$G_0 + x \supseteq 2C_4 \text{ for each } x \in V(G) - V(G_0) \text{ with } e(x, G_0) \geq 2. \tag{36}$$

By a direct verification, we see that (35) holds. To observe (36), we see that if $e(x, Q_2) \geq 2$ then $x \to (Q_2, a_i; V(T))$ for some $a_i \in V(Q_2)$ and obviously, if $e(x, T) \geq 2$ then $G_0 + x \supseteq 2C_4$. Moreover, if $e(x, Q_2) = 1$ and $e(x, T) = 1$ then $[T + x, Q_2] \supseteq 2C_4$ by Lemma 3.4(b). By (36), $e(c_i, G_0) \leq 1$ for each $c_i \in V(Q_1)$ as $G_2 \not\supseteq 3C_4$ and so $e(c_i, Q_2) = 0$ for each $c_i \in N(x_1x_2, Q_1)$. Furthermore, if $c_ix_0 \in E$ with $c_i \in V(Q_1)$ then $e(c_i, z_1z_2) = 0$ for otherwise $x_2 \to (Q_1; x_0x_1z_r)$ for some $r \in$

$\{1,2\}$ and so $G_2 \supseteq 3C_4$ by the second formula (35). Hence $e(z_1z_2, Q_1) = 0$. Thus if $F_1 = x_0x_1z_1z_2x_1$, then

$$e(F_1 - x_1, G_2) \leq 17 \text{ and } e(F_1 - x_1, G_2) + e(c_4, G_2) \leq 22. \tag{37}$$

Lemma 4.12 *Claim 2.3 holds and there exists Q_p in H_1 such that $e(x_1x_3, Q_p) = 7+q$ and $e(T, Q_p) \geq 10 - q$ for some $q \in \{0,1\}$.*

Proof. On the contrary, suppose that the lemma fails. If Claim 2.3 fails, then by the assumption on Q_1, we have that $\tau(Q_1) = 2$, $e(x_0, Q_1) = 4$, $x_1c_2 \in E$, $e(x_2, c_3c_4) = 2$ and $e(x_3, Q_1) = 0$. Clearly, $c_2 \in \mathcal{T}$ as $x_0 \Rightarrow (Q_1, c_2)$. If Claim 2.3 holds but there exists no Q_p in H_1 such that $e(x_1x_3, Q_p) = 7 + q$ and $e(T, Q_p) \geq 10 - q$ for some $q \in \{0,1\}$, then by Subclaim (b), there exists Q_p in H_2, say $Q_p = Q_3$, such that one of (1^0) to (3^0) holds w.r.t. $Q_p = Q_3$. Thus there exists $v_0 \in N(x_1, Q_3)$ such that either $x_0 \Rightarrow (Q_3, v_0)$ or $c_4 \Rightarrow (Q_3, v_0)$ and so $v_0 \in \mathcal{T}$. Furthermore, as $e(c_4, Q_3) \geq 3$ we have $c_4 \to Q_3$. For convenience, we define $v_0 = c_2$ if Claim 2.3 fails. Thus in any case, there exists a strong feasible chain σ_1 such that v_0x_1 and T are the first two items of σ_1 and v_0 is its terminal point. Moreover, each Q_i in $\mathcal{Q} - \{Q_1, Q_3\}$ is still an item of σ_1 and if $v_0 = c_2$ then Q_3 is an item of σ_1 as well. Let $F_2 = T + v_0x_1$ and $R = V(F_1 - x_1) \cup \{v_0\}$.

As $[F_2, Q_2] \not\supseteq 2C_4$ and by (36), $e(v_0, G_0) \leq 1$. Thus if $v_0 \in V(Q_1)$ (i.e., $v_0 = c_2$) then $e(v_0, G_2) \leq 5$ and if $v_0 \in V(Q_3)$ then $e(v_0, G_3) \leq 9$. Together with (37), we see that if $v_0 \in V(Q_1)$ then $e(R, G_2) \leq 22$. We claim that if $v_0 \in V(Q_3)$ then $e(R, G_3) \leq 30$. To see this, we have that $x_2 \to (Q_1, c_4)$ and $c_4 \to Q_3$. As $G_3 \not\supseteq 4C_4$ and by the second formula of (35), $[y, x_0, x_1, z_i] \not\supseteq C_4$ for all $y \in V(Q_3)$ and $i \in \{1,2\}$. Thus $i(x_0z_i, Q_3) = 0$ for all $i \in \{1,2\}$. Moreover, we shall have $e(y, G_0) \leq 1$ for all $y \in V(Q_3)$ by (36). It follows that $e(x_0z_1z_2, Q_3) \leq 4$. With (37), we obtain that $e(R, G_3) \leq 17 + 4 + e(v_0, G_3) \leq 30$. Thus $e(R, H_2) \geq 8(k-3) + 2$ if $v_0 \in V(Q_1)$ and $e(R, H_3) \geq 8(k-4) + 2$ if $v_0 \in V(Q_3)$. Therefore there exists Q_r in H_2 such that $e(R, Q_r) \geq 9$ and if $v_0 \in V(Q_3)$ then $r \geq 4$.

By the first formula of (35), we see that $[u, z_1, z_2, x_1, Q_r] \not\supseteq 2C_4$ for each $u \in \{x_0, v_0\}$ for otherwise either $[F, Q_2, Q_r] \supseteq 3C_4$ or $[F_2, Q_1, Q_2, Q_r] \supseteq 4C_4$. As either $x_2 \to (Q_1, v_0)$ or $x_2 \to (Q_1, c_4)$ and $c_4 \to (Q_3, v_0)$ or $x_0 \to (Q_3, v_0)$, we see that $[x_0, v_0, x_1, z_i, Q_r] \not\supseteq 2C_4$ for each $i \in \{1,2\}$ by the second formula of (35). We conclude that $[Q_r, u, v, x_1, w] \not\supseteq 2C_4$ and so $u \not\to (Q_r; vx_1w)$ for each $\{u, v, w\} \subseteq R$ with $|\{u, v, w\}| = 3$, i.e., $u \not\to (Q_r; R - \{u\})$ for each $u \in R$. As $e(R, Q_r) \geq 9$, it follows that $u \not\to Q_r$ and so $e(u, Q_r) \leq 3$ for all $u \in R$. Moreover, $e(u, Q_r) \leq 2$ for $u \in \{x_0, v_0\}$ by Lemma 4.1(a). Thus $e(z_1z_2, Q_r) \geq 5$ and $e(x_0v_0, Q_r) \geq 3$. W.l.o.g., say $Q_r = d_1d_2d_3d_4d_1$ and $e(z_1, d_1d_2d_3) = 3$. Then $d_2d_4 \notin E$ as $z_1 \not\to Q_r$. Then $e(d_2, R - \{z_1\}) \leq 1$ $e(d_4, R - \{z_1\}) \leq 1$ and so $e(d_2d_4, R) \leq 4$. Similarly, if $e(u, d_2d_4) = 2$ for some $u \in R$

then $e(d_1d_3, R) \leq 4$ and so $e(R, Q_r) \leq 8$, a contradiction. Hence $e(u, d_2d_4) \leq 1$ for all $u \in R$. We claim $z_2d_2 \notin E$. If this is false, say $z_2d_2 \in E$. Then for each $u \in \{x_0, v_0\}$, $e(u, d_1d_3) \leq 1$ as $u \not\to (Q_r; z_1z_2)$. Moreover, $e(d_2, x_0v_0) = 0$ as $z_1 \not\to (Q_r; z_2u)$ for each $u \in \{x_0, v_0\}$. As $e(d_4, x_0v_0) \leq 1$, it follows that $e(z_2, d_1d_2d_3) = 3$, $e(d_4, x_0v_0) = 1$ and $e(x_0v_0, d_1d_3) = 2$. Let $\{u, w\} = \{x_0, v_0\}$ be such that $e(u, Q_r) = 2$. Then $ud_4 \in E$, $e(u, d_1d_3) = 1$ and $e(w, d_1d_3) = 1$. W.l.o.g., say $ud_1 \in E$. Then $[u, d_1, d_4] \supseteq C_3$ and $[z_1, z_2, d_2, d_3] \cong K_4 \geq Q_2$. As $G_0 - \{z_1, z_2\} \supseteq C_4^+$, we shall have $\tau(Q_r) \geq 1$ by (1). Thus $d_1d_3 \in E$ and so $u \to (Q_r; z_1z_2)$, a contradiction. Hence $z_2d_2 \notin E$. Thus $e(z_2, d_1d_3) \geq 1$. W.l.o.g., say $z_2d_3 \in E$. Thus $G_0 + d_3 \supseteq 2C_4$. By Corollary 4.9.1(a), $e(u, d_1d_3) \leq 1$ for $u \in \{x_0, v_0\}$. As $e(d_i, R - \{z_1\}) \leq 1$ for $i \in \{2, 4\}$, we obtain that $4 \geq e(R - \{z_1\}, d_1d_3) \geq 9 - 3 - e(d_2d_4, R - \{z_1\}) \geq 4$. It follows that $e(z_2, d_1d_3) = 2$ and $e(d_2, R - \{z_1\}) = 1$. Thus $z_2 \to (Q_r, d_2; R - \{z_2\})$, a contradiction. □

By Lemma 4.12, there exists Q_p in H_1 such that $e(x_1x_3, Q_p) = 7 + q$ and $e(T, Q_p) \geq 10 - q$ for some $q \in \{0, 1\}$. Clearly, $e(x_3, Q_p) + e(T, Q_p) \geq 17 - e(x_1, Q_p) \geq 13$. By our assumption on Q_2, $e(T, Q_2) \geq e(T, Q_p)$. Therefore if $e(x_1x_3, Q_2) = 7$ then $e(T, Q_2) \geq 10$ and so $e(x_2, Q_2) \geq 3$. Thus if $e(x_1x_3, Q_2) = 7$ then $\tau(Q_2) = 2$ by Lemma 4.1(b). Hence both (29) and (30) do not hold. If $e(x_1, Q_2) = 3$ and $N(x_2, Q_2) = N(x_1, Q_2)$ then (31) holds with $x_2a_3 \in E$ and if $e(x_1, Q_2) = 3$ and $N(x_2, Q_2) \neq N(x_1, Q_2)$ then we may assume that (32) holds with $x_2a_2 \in E$. Let $R_1 = \{x_0, z_1, z_2, c_4\}$. By (37), $e(R_1, G_2) \leq 22$ and so $e(R_1, H_2) \geq 8k - 22 = 8(k-3) + 2$. Say $e(R_1, Q_3) \geq 9$. The next lemma will complete the proof of Claim 2.4.

Lemma 4.13 *There exists a labelling $Q_3 = d_1d_2d_3d_4d_1$ such that $e(R_1, Q_3) = 9$, $e(z_1z_2, d_2d_3d_4) = 6$ and $d_3c_4 \in E$.*

Proof. As $G_3 \not\supseteq 4C_4$ and by the first formula of (35), we have (38) below. Since $x_i \to (Q_1, c_4)$ for $i \in \{0, 2\}$ and by the first and second formulas of (35), we have (39) below:

$$u \not\to (Q_3; vx_1w), \text{i.e., } u \not\to (Q_3; vw), \text{ for each permutation } (u, v, w) \text{ of } \{x_0, z_1, z_2\}; \quad (38)$$

$$c_4 \not\to (Q_3; ux_1v) \text{ i.e., } c_4 \not\to (Q_3; uv), \text{ for each } \{u, v\} \subseteq \{x_0, z_1, z_2\} \text{ with } u \neq v. \quad (39)$$

Let $Q_3 = d_1d_2d_3d_4d_1$. As $e(R_1, Q_3) \geq 9$ and by (39), $c_4 \not\to Q_3$. By Lemma 4.1(a), $e(c_4, Q_3) \leq 2$. We shall show that $e(x_0, Q_3) \leq 2$. Suppose that $e(x_0, Q_3) = 4$. Then $\tau(Q_3) = 2$ by Lemma 4.1(a). As $e(c_4, Q_3) \leq 2$, $e(z_1z_2, Q_3) \geq 3$. W.l.o.g., say $e(z_1, Q_3) \geq 2$. As $x_0 \not\to (Q_3; z_1z_2)$, $i(z_1z_2, Q_3) = 0$. If $e(z_2, Q_3) \geq 1$, then $z_1 \to (Q_3; x_0z_2)$, a contradiction. Hence $e(z_2, Q_3) = 0$ and so $e(z_1, Q_3) \geq 3$. As $c_4 \not\to (Q_3; x_0z_1)$, $e(c_4, Q_3) \leq 1$. It follows that $e(z_1, Q_3) = 4$ and $e(c_4, Q_3) = 1$, contradicting Lemma 4.10. Next, suppose $e(x_0, Q_3) = 3$. Say $e(x_0, d_1d_2d_3) = 3$. Then

$e(z_1z_2, Q_3) \geq 4$. By Lemma 4.1(a), $x_0 \to Q_3$ with $d_2d_4 \in E$. As $x_0 \not\to (Q_3; z_1z_2)$, it follows that $e(d_i, z_1z_2) = 1$ for all $d_i \in V(Q_3)$ and $e(c_4, Q_3) = 2$. Thus $c_4 \to (Q_3; x_0z_i)$ for some $i \in \{1, 2\}$, a contradiction.

Suppose $e(x_0, Q_3) = 0$. If $e(c_4, Q_3) = 1$ then $e(z_1z_2, Q_3) = 8$ and so the lemma holds. So assume $e(c_4, Q_3) = 2$. Then $N(c_4, Q_3) = \{d_i, d_{i+1}\}$ for some $i \in \{1, 2, 3, 4\}$ since $c_4 \not\to (Q_3; z_1z_2)$. Say w.l.o.g. $N(c_4, Q_3) = \{d_3, d_4\}$. If $e(d_1d_2, z_1z_2) = 4$, we have that $[d_1, d_2, z_1, z_2] \cong K_4 \geq Q_2$, $[c_4, d_3, d_4] \supseteq C_3$ and $G_0 - \{z_1, z_2\} \supseteq C_4^+$. By (1), $\tau(Q_3) \geq 1$ and so $c_4 \to (Q_3; z_1z_2)$, a contradiction. Hence $e(d_1d_2, z_1z_2) \leq 3$. W.l.o.g., say $e(d_1, z_1z_2) \leq 1$. It follows that $e(R_1, Q_3) = 9$ with $e(z_1z_2, d_2d_3d_4) = 6$ and so the lemma holds. Therefore we may assume that $1 \leq e(x_0, Q_3) \leq 2$ in the following. Note that $e(F_1 - x_1, Q_3) \geq 9 - e(c_4, Q_3) \geq 7$.

Let Q_2' be a 4-cycle of $G_0 - V(T_1)$ where $T_1 = x_1z_1z_2x_1$. Suppose that $e(T, Q_2) \geq 11$ or one of (27), (32), (33) and (34) holds. Recall that $x_2a_2 \in E$ when (32) holds as assumed. In each of these cases, $\tau(Q_2') = \tau(Q_2)$. Thus we may apply Lemmas 4.4-4.6 to F_1 and Q_3 and see that (9) holds w.r.t. F_1 and Q_3. By Lemma 4.2, $[F_1, Q_3] \not\supseteq P \uplus Q$ with $P \supseteq 2P_2$, $Q \cong C_4$ and $\tau(Q) = \tau(Q_3) + 2$. Then we apply Lemma 3.3 to F_1, Q_3 and c_4 and see that the lemma holds.

Therefore we may assume that $e(T, Q_2) \leq 10$, $\tau(Q_2') < \tau(Q_2)$ and either (28) or (31) holds in the remaining proof. We note two observations here. *Observation A*: For each $u \in \{x_0, c_4\}$, $[u, z_1, z_2, Q_3] \not\supseteq C$ with $C \cong C_3$ such that $[u, z_1, z_2, Q_3] - V(C) > Q_3$. We see this by (1) since $[x_1, x_2, x_3, a_1] \cong K_4 \geq Q_2$. *Observation B*: $[x_0, c_4, z_1, z_2, Q_3] \not\supseteq 2C_4$. We see this since $x_2 \to (Q_1, c_4)$ and $G_0 - \{z_1, z_2, x_2\} \supseteq C_4$ by (35).

We will apply Corollary 4.9.1 and Lemma 4.11 to either F, Q_2 and Q_3 or F', Q_2 and Q_3. Note that $e(x_1x_2, Q_2) \leq 6$ and $e(x_3, Q_2) = 4$.

As $e(z_1z_2, Q_3) \geq 9 - e(x_0c_4, Q_3) \geq 5$, say w.l.o.g. $e(z_1, Q_3) \geq e(z_2, Q_3)$ and $e(z_1, d_1d_2d_3) = 3$. We claim that $e(u, d_1d_3) \leq 1$ and $e(u, d_2d_4) \leq 1$ for each $u \in \{x_0, c_4\}$ and $e(z_2, d_2d_4) \leq 1$. By (38), $e(d_i, x_0z_2) \leq 1$ for each $i \in \{2, 4\}$. If $e(c_4, d_2d_4) = 2$ then $e(x_0z_2, d_1d_3) = 0$ by (39) and it follows that $e(R_1, Q_3) \leq 8$, a contradiction. If $e(u, d_2d_4) = 2$ for some $u \in \{x_0, z_2\}$, then $e(w, d_2d_4) = 0$ where $\{u, w\} = \{x_0, z_2\}$. Moreover, as $u \not\to (Q_3; z_1w)$ by (38), we have $e(w, d_1d_3) = 0$. As $1 \leq e(x_0, Q_3) \leq 2$, we obtain $u = x_0$ and so $e(F_1 - x_1, Q_3) \leq 6$, a contradiction. Suppose that $e(u, d_1d_3) = 2$ for some $u \in \{x_0, c_4\}$ then $z_2d_2 \notin E$ and $e(d_4, z_1z_2) \leq 1$ as $u \not\to (Q_3; z_1z_2)$. Thus $e(z_2, d_1d_3) \geq 1$ as $e(z_1z_2, Q_3) \geq 5$. Say w.l.o.g. $z_2d_3 \in E$. Then $G_0 + d_3 \supseteq 2C_4$. If $d_2d_4 \notin E$, we obtain a contradiction with Corollary 4.9.1(a). Hence $d_2d_4 \in E$. Thus $z_1 \to Q_3$. As $z_1 \not\to (Q_3; x_0z_2)$, $i(x_0z_2, Q_3) = 0$ and so $u = c_4$. As $c_4 \not\to (Q_3; R_1 - \{c_4\})$, $e(d_4, x_0z_1z_2) \leq 1$ and $e(d_2, x_0z_2) = 0$. As $e(F_1 - x_1, Q_3) \geq 7$, $e(x_0z_2, d_1d_3) \geq 3$ and so $i(x_0z_2, d_1d_3) \geq 1$, a contradiction. Hence the claim holds.

Suppose that $e(z_1, Q_3) = 3$. Then $3 \geq e(z_2, Q_3) \geq 2$, $e(x_0z_2, Q_3) \geq 4$ and $e(x_0c_4, Q_3) \geq 3$. If $d_2d_4 \in E$ then $z_1 \to Q_3$. By (38), $e(d_i, x_0z_2) = 1$ for all

$d_i \in V(Q_3)$. Thus $e(c_4, Q_3) = 2$ and $c_4 \to (Q_3; R_1 - \{c_4\})$, a contradiction. Therefore $d_2d_4 \notin E$. Assume that $e(u, d_id_4) = 2$ for some $i \in \{1, 3\}$ and $u \in \{x_0, c_4\}$. Say w.l.o.g. $e(u, d_1d_4) = 2$. If $e(z_2, d_2d_3) \geq 1$ then $[u, d_1, d_4] \supseteq C_3$ and $[z_1, z_2, d_2, d_3] \supseteq C_4^+$. By Observation A, $\tau(Q_3) = 1$ and so $d_1d_3 \in E$. This contradicts Lemma 4.11. Hence $e(z_2, d_2d_3) = 0$ and so $e(z_2, d_1d_4) = 2$. Since $z_1 \not\to (Q_3, d_4; x_0z_2)$, $x_0d_4 \notin E$ and so $u = c_4$. As $[z_1, d_2, d_3] \cong C_3$ and $[z_2, d_1, d_4, c_4] \supseteq C_4^+$, we obtain $d_1d_3 \in E$ by Observation A. Then $x_0d_2 \notin E$ as $c_4 \not\to (Q_3; x_0z_1)$. Consequently, $e(x_0, d_1d_3) = 2$, a contradiction. Hence $e(u, d_id_4) \neq 2$ for each $i \in \{1, 3\}$ and $u \in \{x_0, c_4\}$. Assume that $e(x_0, Q_3) = 2$. Then $e(x_0, d_2d_i) = 2$ for some $i \in \{1, 3\}$. Say w.l.o.g. $e(x_0, d_1d_2) = 2$. Then $z_2d_2 \notin E$ as $z_1 \not\to (Q_3; x_0z_2)$ and $e(z_2, d_1d_3) \leq 1$ as $z_2 \not\to (Q_3; x_0z_1)$. It follows that $z_2d_4 \in E$, $e(z_2, d_1d_3) = 1$ and $e(c_4, Q_3) = 2$. If $e(c_4, d_1d_2) = 2$ then $[c_4, d_1, x_0, d_2] \supseteq C_4$ and $[z_1, d_3, d_4, z_2] \supseteq C_4$, contradicting Observation B. Hence $e(c_4, d_2d_3) = 2$. If $z_2d_1 \in E$ then $[z_2, d_1, d_4] \cong C_3$ and $[z_1, d_2, c_4, d_3] \supseteq C_4^+$ and if $z_2d_3 \in E$ then $[z_2, d_3, d_4] \cong C_3$ and $[x_0, d_1, z_1, d_2] \supseteq C_4^+$. By Observation A, $d_1d_3 \in E$. Thus $z_2 \to (Q_3; x_0z_1)$, a contradiction. We conclude that $e(x_0, Q_3) = 1$. It follows that $e(z_2, Q_3) = 3$ and $e(c_4, Q_3) = 2$. Then $e(c_4, d_2d_i) = 2$ for some $i \in \{1, 3\}$. W.l.o.g., say $e(c_4, d_1d_2) = 2$. If $z_2d_4 \in E$ then $z_2d_2 \notin E$ as $e(z_2, d_2d_4) \leq 1$. Thus $e(z_2, d_1d_4d_3) = 3$. Then $[c_4, d_1, d_2] \supseteq C_3$ and $[z_1, z_2, d_3, d_4] \supseteq C_4^+$. By Observation A, $d_1d_3 \in E$. This contradicts Lemma 4.11. Hence $z_2d_4 \notin E$. Thus $e(z_2, d_1d_2d_3) = 3$. By renaming d_i as d_{i+1} for all $d_i \in V(Q_3)$, we see that Lemma 4.13 holds.

Finally, $e(z_1, Q_3) = 4$. Assume $e(x_0, Q_3) = 2$. Say w.l.o.g. $e(x_0, d_1d_4) = 2$. Then $e(z_2, d_1d_4) = 0$ as $z_1 \not\to (Q_3; x_0z_2)$. Thus $e(z_2, d_2d_3) \geq 1$. W.l.o.g., say $z_2d_3 \in E$. Then $d_2d_4 \notin E$ as $x_0 \not\to (Q_3; z_1z_2)$. As $[z_1, z_2, d_2, d_3] \supseteq C_4^+$ and $[x_0, d_1, d_4] \cong C_3$, we get $d_1d_3 \in E$ by Observation A. This contradicts Lemma 4.11. Hence $e(x_0, Q_3) = 1$. Say $x_0d_1 \in E$. Then $z_2d_1 \notin E$. As $e(z_2, d_2d_4) \leq 1$, it follows that $e(z_2, Q_3) = 2$ and $e(c_4, Q_3) = 2$. W.l.o.g., say $e(z_2, d_2d_3) = 2$. As $z_2 \not\to (Q_3; x_0z_1)$, $d_2d_4 \notin E$. Assume that $e(c_4, d_id_{i+1}) = 2$ for some $i \in \{1, 3, 4\}$, i.e., $e(c_4, d_2d_3) \neq 2$. Then $[c_4, d_i, d_{i+1}] \supseteq C_3$ and $[z_1, z_2, d_{i+2}, d_{i+3}] \supseteq C_4^+$. By Observation A, $d_1d_3 \in E$. This contradicts Lemma 4.11 (if necessary, exchanging the subscripts of d_1 with d_3 or d_2 with d_4). Therefore $e(c_4, d_2d_3) = 2$. Then $[z_1, d_1, d_4] \cong C_3$ and $[z_2, d_2, c_4, d_4] \supseteq C_4^+$. By observation A, $d_1d_3 \in E$.

Let $S = V(F) \cup \{c_4, d_4\}$. As $c_4 \Rightarrow (Q_3, d_4)$, $d_4 \in \mathcal{T}$ and by (36), $e(d_4, G_0) = 1$. As $e(x_0, Q_3) = 1$ and $[F, Q_3] \not\supseteq 2C_4$, we have $e(F, Q_3) \leq 9$ by Claim 2.2. As $e(T, Q_2) \leq 10$ and $e(F, G_1) \leq 16$, we get $e(F, G_3) \leq 35$. Clearly, $e(c_4, G_3) \leq 7$. As $x_3 \to (Q_2, z_1)$ and $z_1 \to (Q_3, d_4)$, we have $d_4 \not\to (Q_1; x_0x_1x_2)$. This implies that $e(d_4, Q_1) \leq 1$. Thus $e(d_4, G_3) \leq 4$. Hence $e(S, G_3) \leq 35 + 7 + 4 = 46$ and so $e(S, H_3) \geq 12k - 46 = 12(k-4) + 2$. Say $e(S, Q_4) \geq 13$. If $e(F, Q_4) \geq 9$, then we see, by Claim 2.2, that $e(x_0, Q_4) = 0$ for otherwise $e(F, Q_4) = 9$, $[T, Q_4, w] \supseteq 2C_4$ where $w \in \{c_4, d_4\}$ with $e(w, Q_4) \geq 2$ and so $[F, Q_1, Q_3, Q_4] \supseteq 4C_4$. Thus $e(T, Q_4) \geq 9$. As $x_0 \Rightarrow (Q_1, c_4)$ and

$c_4 \Rightarrow (Q_3, d_4)$, we see that $e(w, Q_4) \leq 1$ for each $w \in \{c_4, d_4\}$ by Lemma 3.2. Thus $e(T, Q_4) \geq 11$ and so $e(T, Q_2) \geq 11$ by the assumption on Q_2, a contradiction. Hence $e(F, Q_4) \leq 8$ and so $e(c_4d_4, Q_4) \geq 5$. Let $w \in \{c_4, d_4\}$ be such that $e(w, Q_4) \geq 3$. Then $w \to Q_4$ by Lemma 4.1(a). Hence $e(y, T) \leq 1$ for all $y \in V(Q_4)$ for otherwise $[F, Q_1, Q_3, Q_4] \supseteq 4C_4$. Thus $e(x_0c_4d_4, Q_4) \geq 9$. Then $e(x_0, Q_4) \geq 3$ or $e(c_4, Q_4) \geq 3$. If $e(x_0, Q_4) \geq 3$ then $x_0 \to (Q_4; c_4d_3d_4)$, $[z_1, z_2, d_1, d_2] \supseteq C_4$ and $x_2 \to (Q_1, c_4)$. If $e(c_4, Q_4) \geq 3$ then $c_4 \to (Q_4; x_0d_1d_4)$, $[z_1, z_2, d_2, d_3] \supseteq C_4$ and $x_2 \to (Q_1, c_4)$. As $G_0 - \{z_1, z_2, x_2\} \supseteq C_4$, we obtain $G_4 \supseteq 5C_4$, a contradiction. This proves the lemma. □

By Lemma 4.13, we see that $e(x_2, z_1z_2) = 0$, for if $e(x_2, z_1z_2) \geq 1$, say $x_2z_2 \in E$, then $z_1 \to (Q_3, d_3; c_4x_2z_2)$ and so $G_3 \supseteq 4C_4$ since $G_0 - \{z_1, z_2, x_2\} \supseteq C_4$ by (35). Thus $e(T, Q_2) \leq 10$ and each of (29) to (34) does not hold. Hence (27) or (28) holds. Then $\{a_2, a_3\}$ and $\{a_3, a_4\}$ are in the symmetric position for $\{z_1, z_2\}$. Therefore as obtaining (37), we also have $e(x_0c_4a_2a_3, G_2) \leq 22$ and if $e(x_0c_4a_2a_3, Q_3) \geq 9$ then as above, $e(x_0c_4a_2a_3, Q_3) = 9$. Hence $e(x_0c_4a_2a_3, Q_3) \leq 9$. Thus $e(x_0c_4a_2a_3, H_3) \geq 8k - 22 - 9 = 8(k-4) + 1$. Say $e(x_0c_4a_2a_3, Q_4) \geq 9$. By Lemma 4.13, there exists a labelling $Q_4 = u_1u_2u_3u_4u_1$ such that $e(a_2a_3, u_2u_3u_4) = 6$ and $c_4u_3 \in E$. Thus $[a_3, d_3, c_4, u_3] \supseteq C_4$, $a_4 \to (Q_3, d_3)$, $a_2 \to (Q_4, u_3)$, $[T, a_1] \supseteq C_4$ and $x_0 \to (Q_1, c_4)$, i.e., $G_4 \supseteq 5C_4$, a contradiction. ∎

Proof of Claim 2.5. Suppose that the claim is false. By Lemmas 4.4-4.6 and Claim 2.4, we may assume that (12) holds. Say $Q = Q_1 = c_1c_2c_3c_4c_1, N(x_0, Q_1) = N(x_2, Q_1) = \{c_1, c_2, c_3\}$, $x_3c_4 \in E$ and $c_2c_4 \in E$. Let $F' = T + c_4x_3$. Clearly, G_1 has an automorphism f such that $f(F) = F'$ and $f(c_i) = c_i$ for $i \in \{1, 2, 3\}$. As $G_1 \not\supseteq 2C_4$, $e(x_1, Q_1) = 0$. Then $e(F + c_4, G_1) = 19$ and so $e(F + c_4, H_1) \geq 10(k-2) + 1$. Say $e(F + c_4, Q_2) \geq 11$. First, assume $e(u, Q_2) \geq 3$ for some $u \in \{x_0, c_4\}$. W.l.o.g., say $e(x_0, Q_2) \geq 3$. Then $x_0 \to Q_2$. Thus $e(v, T) \leq 1$ for all $v \in V(Q_2)$. Hence $e(x_0c_4, Q_2) \geq 7$. W.l.o.g., say $e(x_0, Q_2) = 4$. Then $\tau(Q_2) = 2$ by Lemma 4.1(a). As $x_2 \to (Q_1, c_4)$, $c_4 \not\to (Q_2; x_0x_1x_3)$ and so $e(x_3, Q_2) = 0$. By Claim 2.4, $e(x_0x_2, Q_2) \leq 6$ and so $e(c_4x_1, Q_2) \geq 5$. It follows that $x_0 \to (Q_2; c_4x_3x_1)$ and so $G_2 \supseteq 3C_4$, a contradiction. Hence $e(u, Q_2) \leq 2$ for each $u \in \{x_0, c_4\}$. Thus $e(F, Q_2) \geq 9$. By Claim 2.2, we see that $e(x_0, Q_2) = 0$ for otherwise $e(F, Q_2) = 9$, $e(c_4, Q_2) = 2$, $[T, Q_2, c_4] \supseteq 2C_4$ and so $G_2 \supseteq 3C_4$. Thus $e(F', Q_2) \geq 11$. By Claim 2.2, $e(c_4, Q_2) = 0$ and so $e(T, Q_2) \geq 11$. By Lemma 4.1(b), $\tau(Q_2) = 2$. By Lemma 3.1(c), we may label $Q_2 = b_1b_2b_3b_4b_1$ such that $e(x_1, b_1b_2) = 2$ and $[x_2, x_3, b_3, b_4] \cong K_4$. Say $F_1 = x_0x_1b_1b_2x_1$ and $Q_2' = x_2x_3b_3b_4x_2$. Then $\sigma_1 = (x_0x_1, x_1b_1b_2x_1, Q_1, Q_2', Q_3, \ldots, Q_{k-1})$ is a strong feasible chain. As $x_0 \to Q_1$, $[T, Q_2, c_i] \not\supseteq 2C_4$ and so $e(c_i, Q_2) = 0$ for all $c_i \in V(Q_1)$. Thus $e(S, G_2) \leq 20$ where $S = \{x_0, c_4, b_1, b_2\}$. Hence $e(S, H_2) \geq 8k - 20 = 8(k-3) + 4$. Say $e(S, Q_3) \geq 9$. As $x_i \to Q_1$ for $i \in \{0, 2\}$, we readily see that $c_4 \not\to (Q_3; ux_1v)$ for each $\{u,v\} \subseteq \{x_0,b_1,b_2\}$ with $u \neq v$ for otherwise $G_3 \supseteq 4C_4$. As $e(S,Q_3) \geq 9$, this implies that $c_4 \not\to Q_3$. By Lemma 4.1(a), $e(c_4,Q_3) \leq 2$. Hence $e(F_1 - x_1, Q_3) \geq 7$. By Lemmas 4.4-4.6 and Claim 2.4, either $e(x_0,Q_3) = 0$ or one of (9) and (12) holds w.r.t. F_1 and Q_3. However, if (12) holds w.r.t. F_1 and Q_3, then $e(c_4,Q_3) \geq 2$ and so $c_4 \to (Q_3; x_0x_1b_i)$ where $i \in \{1,2\}$ with $N(b_i,Q_3) = N(x_0,Q_3)$, a contradiction. Hence $e(x_0,Q_3) = 0$ or (9) holds w.r.t. F_1 and Q_3. By Lemma 4.2, $[F_1,Q_3] \not\supseteq P \uplus Q$ with $P \supseteq 2P_2$, $Q \cong C_4$ and $\tau(Q) = \tau(Q_3)+2$. Applying Lemma 3.3 to F_1, Q_3 and c_4, there exists a labelling $Q_3 = d_1d_2d_3d_4d_1$ such that $e(b_1b_2, d_2d_3d_4) = 3$ and $c_4d_3 \in E$. As $e(x_3,Q_2) \geq 3$, $e(x_3,b_1b_2) \geq 1$. Say w.l.o.g. $x_3b_2 \in E$. Then $[c_4,x_3,b_2,d_3] \supseteq C_4$, $b_1 \to (Q_3,d_3)$, $[x_1,x_2,b_3,b_4] \supseteq C_4$ and $x_0 \to (Q_1,c_4)$, i.e., $G_3 \supseteq 4C_4$, a contradiction ∎

Lemma 4.14 *Let $\{i,r\} \subseteq \{1,\dots,k-1\}$ with $i \neq r$ and $z \in V(Q_i)$. Suppose that $e(F+z,Q_r) \geq 11$ and $e(z,x_2x_3) = 1$. Furthermore, suppose that either $x_0 \Rightarrow (Q_i,z)$ or there exists Q_j with $j \neq i,r$ such that $x_0 \Rightarrow (Q_j,y)$ and $y \Rightarrow (Q_r,z)$ for some $y \in V(Q_j)$. Then $e(x_0z,Q_r) = 0$ and so $e(T,Q_r) \geq 11$.*

Proof. For convenience, say $Q_i = Q_1$, $Q_r = Q_2$ and $x_2z \in E$. Moreover, if $x_0 \not\Rightarrow (Q_1,z)$, say $Q_j = Q_3$. If $x_0 \Rightarrow (Q_1,z)$, let $[Q_1 - z + x_0] \supseteq Q' \cong C_4$. If $x_0 \not\Rightarrow (Q_1,z)$, let $[Q_3 - y + x_0] \supseteq Q' \cong C_4$ and $[Q_1 - z + y] \supseteq Q'' \cong C_4$. Then $\sigma' = (zx_2, T, Q', Q_2, \dots, Q_{k-1})$ is a strong feasible chain if $x_0 \Rightarrow (Q_1,z)$ and otherwise $\sigma' = (zx_2, T, Q', Q'', Q_2, Q_4, \dots, Q_{k-1})$ is a strong feasible chain. Say $F' = T + zx_2$. If $e(F',Q_2) \geq 9$, then by Claim 2.2, we see that $e(z,Q_2) = 0$ for otherwise $e(x_0,Q_2) \geq 2$ and $[T,x_0,Q_2] \supseteq 2C_4$. Consequently, $e(F,Q_2) \geq 11$ and so $e(x_0,Q_2) = 0$ by Claim 2.2. Thus the lemma holds. Hence assume $e(F',Q_2) \leq 8$. Then $e(x_0,Q_2) \geq 3$ and so $x_0 \to Q_2$. Thus $e(u,T) \leq 1$ for all $u \in V(Q_2)$. Hence $8 \geq e(zx_0,Q_2) \geq 7$ and $4 \geq e(T,Q_2) \geq 3$. As either $e(x_0,Q_2) = 4$ or $e(z,Q_2) = 4$, we have $\tau(Q_2) = 2$ by Lemma 4.1(a). As the roles of F and F' can be exchanged in the following argument, we may assume w.l.o.g. that $e(x_0,Q_2) = 4$. Suppose $e(x_3,Q_2) = 0$. By Claim 2.4, $e(x_0x_2,Q_2) \leq 6$ and so $e(x_1,Q_2) \geq 5 - e(c_4,Q_2) \geq 1$. Similarly, $e(zx_1,Q_2) \leq 6$ and so $e(x_2,Q_2) \geq 1$. Applying Claim 2.3 to F and Q_2, we get $e(x_2,Q_2) = 1$. Thus $e(x_1,Q_2) \geq 2$. Then applying Claim 2.3 to F' and Q_2, we see that $e(z,Q_2) \neq 4$. It follows that $e(z,Q_2) = e(x_1,Q_2) = 3$. Let $x' \in N(x_2,Q_2)$ and $[Q_2 - x' + x_0] \supseteq Q_2' \cong C_4$. Then $\tau(Q_2') = \tau(Q_2)$ and $e(x'x_1,Q_2') = 8$. This contradicts Claim 2.4 since $(x'x_2, T, Q_1, Q_2', Q_3, \dots, Q_{k-1})$ is a strong feasible chain. Therefore $e(x_3,Q_2) \geq 1$. By Claim 2.5, $e(F - x_1, Q_2) \leq 6$. Thus $e(x_2x_3,Q_2) \leq 2$. Suppose $e(x_3,Q_2) = 2$. Then $e(x_2,Q_2) = 0$. Applying Claim 2.3 to F and Q_2, we get $e(x_1,Q_2) = 0$. Thus $e(T,Q_2) \leq 2$, a contradiction. Hence $e(x_3,Q_2) = 1$ and $e(x_2,Q_2) \leq 1$. Then $e(x_1,Q_2) \geq 1$ as $e(T,Q_2) \geq 3$. As $e(z,Q_2) \geq 3$ and by Claim 2.5, $e(F' - x_2, Q_2) \leq 6$. Thus $e(x_2,Q_2) = 1$ since $e(F+z,Q_2) \geq 11$. Since $e(x_0,Q_2) = 4$,

$\tau(Q_2) = 2$ and $e(x_i, Q_2) > 0$ for all $x_i \in V(T)$, we readily see that $[F, Q_2] \supseteq 2C_4$, a contradiction. ∎

Lemma 4.15 *Suppose that $e(T, Q_i) \geq 11$ and $\tau(Q_i) = 2$ for some Q_i in H_1. Let $V(Q_i) = \{b_1, b_2, b_3, b_4\}$ be such that $\{b_1, b_2, b_3\} \subseteq N(x_1)$ and $[x_2, x_3, b_4, b_r] \cong K_4$ for $r = 2, 3$. Furthermore, suppose that Q_1 has a vertex z such that $e(x_0, Q_i) \geq 3$, $x_0 \Rightarrow (Q_1, z)$ and $e(x_0 z b_1 b_r, G_1 \cup Q_i) \leq 22$ for $r = 2, 3$. Then $x_2 \not\to (Q_1, z)$.*

Proof. On the contrary, say $x_2 \to (Q_1, z)$. W.l.o.g., say $Q_i = Q_2$. Let $Q_2' = x_2 x_3 b_3 b_4 x_2$, $T_1 = x_1 b_1 b_2 x_1$, $F_1 = T_1 + x_0 x_1$, $G_0 = [T, Q_2]$ and $S_1 = \{x_0, b_1, b_2, z\}$. Then $\sigma_1 = (x_0 x_1, T_1, Q_1, Q_2', Q_3, \ldots, Q_{k-1})$ is a strong feasible chain. As $e(S_1, G_2) \leq 22$, $e(S_1, H_2) \geq 8(k-3) + 2$. Say $e(S_1, Q_3) \geq 9$. Clearly, $G_0 - \{x_1, b_i, x_2\} \supseteq C_4$ for each $i \in \{1, 2\}$ and $G_0 - \{b_1, x_1, b_2\} \supseteq C_4$. As $x_0 \to (Q_1, z)$ and $x_2 \to (Q_1, z)$, this implies that $z \not\to (Q_3; u x_1 v)$ for each $\{u, v\} \subseteq \{x_0, b_1, b_2\}$ with $u \neq v$ for otherwise $G_3 \supseteq 4C_4$. As $e(S_1, Q_3) \geq 9$, this further implies that $z \not\to Q_3$. By Lemma 4.1(a), $e(z, Q_3) \leq 2$. Thus $e(F_1 - x_1, Q_3) \geq 7$. By Claim 2.5, either $e(x_0, Q_3) = 0$ or (9) holds w.r.t. F_1 and Q_3. By Lemma 4.2, $[F_1, Q_3] \not\supseteq P \uplus Q$ with $P \supseteq 2P_2$, $Q \cong C_4$ and $\tau(Q) = \tau(Q_3) + 2$. By Lemma 3.3, we see that $e(S_1, Q_3) = 9$ and there exists a labelling $Q_3 = d_1 d_2 d_3 d_4 d_1$ such that $e(b_1 b_2, d_2 d_3 d_4) = 6$ and $z d_3 \in E$. Let $S_2 = \{x_0, b_1, b_3, z\}$. Similarly, if $e(S_2, Q_3) \geq 9$ then $e(S_2, Q_3) = 9$. Thus $e(S_2, Q_3) \leq 9$ and so $e(S_2, G_3) \leq 31$. Then $e(S_2, H_3) \geq 8(k-4) + 1$. Say $e(S_2, Q_4) \geq 9$. Similarly, there exists a labelling $Q_4 = a_1 a_2 a_3 a_4 a_1$ such that $e(b_1 b_3, a_2 a_3 a_4) = 6$ and $z a_3 \in E$. It follows that $[z, d_3, b_1, a_3] \supseteq C_4$, $b_2 \to (Q_3, d_3)$, $b_3 \to (Q_4, a_3)$, $T + b_4 \supseteq C_4$ and $x_0 \to (Q_1, z)$, i.e., $[F, Q_1, Q_2, Q_3, Q_4] \supseteq 5C_4$, a contradiction. ∎

Lemma 4.16 *If $e(x_0, Q_1) = 4$ and $e(x_2 x_3, Q_1) \geq 1$ then $e(T, Q_i) \geq 11$ for some Q_i in H_1, $e(x_1, Q_1) = 0$ and $e(x_r, Q_1) \leq 1$ for each $r \in \{2, 3\}$. If $e(x_0, Q_1) = 3$ and $e(x_2 x_3, Q_1) \geq 3$ then $\tau(Q_1) = 2$, $e(T, Q_i) \leq 10$ for all Q_i in H_1, and for some $\{r, t\} = \{2, 3\}$, $e(x_r, Q_1) = 0$ and $N(x_t, Q_1) = N(x_0, Q_1)$.*

Proof. Say $Q_1 = c_1 c_2 c_3 c_4 c_1$. First, suppose that $e(x_0, Q_1) = 4$. By Lemma 4.1(a), $\tau(Q_1) = 2$. Say w.l.o.g. $e(x_2, Q_1) \geq e(x_3, Q_1)$ and $x_2 c_4 \in E$. Let $G_0 = [T, Q_2]$. We show $e(x_2, Q_1) = 1$ first. If this is false, say w.l.o.g. $x_2 c_2 \in E$. Then $e(x_2, Q_1) = 2$ and $e(x_3, Q_1) = 0$ by Claim 2.5. By Claim 2.3, $e(x_1, Q_1) = 0$. Then $e(F + c_4, G_1) = 19$ and so $e(F + c_4, H_1) \geq 10k - 19 = 10(k-2) + 1$. Say $e(F + c_4, Q_2) \geq 11$. By Lemma 4.14, $e(T, Q_2) \geq 11$ and $e(x_0 c_4, Q_2) = 0$. By Lemma 4.1(b), $\tau(Q_2) = 2$. By Lemma 3.1(c), we label $V(Q_2) = \{b_1, b_2, b_3, b_4\}$ such that $\{b_1, b_2, b_3\} \subseteq N(x_1)$, $[x_2, x_3, b_4, b_r] \cong K_4$ for $r = 2, 3$. As $G_2 \not\supseteq 3C_4$ and $x_0 \to Q_1$, $e(c_i, G_0) \leq 1$ for all $c_i \in V(Q_1)$. Hence $e(c_2 c_4, G_0 - x_2) = 0$. If $b_i c_r \in E$ for some $i \in \{1, 2, 3\}$ and $c_r \in \{1, 3\}$ then $x_2 \to (Q_1, c_r; b_i x_1 x_0)$ and $x_3 \to (Q_2, b_i)$, i.e., $G_2 \supseteq 3C_4$, a contradiction. Hence $e(b_1b_2b_3, Q_1) = 0$. It follows that $(x_0b_1b_rc_1, G_2) \leq 22$ for $r = 2, 3$. By Lemma 4.15, $x_2 \not\rightarrow (Q_1, c_1)$, a contradiction. Hence $e(x_2, Q_1) = 1$. If $e(x_1, Q_1) \geq 1$, let $F' = T + c_4x_2$ and $Q_1' = x_0c_1c_2c_3x_0$. Then $\tau(Q_1') = \tau(Q_1)$, $e(c_4, Q_1') = 4$ and $e(x_1, Q_1') \geq 2$. With F' and Q_1' replacing F and Q_1 in this argument, we shall have $e(x_1, Q_1') \leq 1$, a contradiction. Hence $e(x_1, Q_1) = 0$. Then $e(F + c_4, G_1) \leq 19$ and so $e(F + c_4, H_1) \geq 10(k-2) + 1$. Thus $e(F + c_4, Q_i) \geq 11$ for some Q_i in H_1 and so $e(T, Q_i) \geq 11$ by Lemma 4.14.

Next, suppose that $e(x_0, Q_1) = 3$. By Claim 2.5, $e(x_2x_3, Q_1) = 3$. Say $e(x_0, c_1c_2c_3) = 3$. Then $c_2c_4 \in E$. Say w.l.o.g. $e(x_2, Q_1) \geq e(x_3, Q_1)$. Suppose that $e(Q_i, T) \geq 11$ for some Q_i in H_1. We may assume that $e(T, Q_2) \geq 11$. Let Q_2 be labelled and G_0 defined as above. Then for each $c_i \in V(Q_1)$, $G_0 + c_i \not\supseteq 2C_4$ and so $e(c_i, G_0) \leq 1$. Thus $e(c_i, G_2) \leq 5$ for all $c_i \in V(Q_1)$ and $e(c_i, G_0 - \{x_2, x_3\}) = 0$ for each $c_i \in N(x_2x_3, Q_1)$. Hence $e(b_1b_2b_3, Q_1) \leq 1$. Thus $e(b_1b_i, G_2) \leq 13$ and so $e(x_0b_1b_ic_j, G_2) \leq 22$ for each $i \in \{2, 3\}$ and $c_j \in V(Q_1)$. By Lemma 4.15, $x_2 \not\rightarrow (Q_1, c_j)$ for each c_j with $x_0 \Rightarrow (Q_1, c_j)$. As $x_0 \Rightarrow (Q_1, c_4)$, $x_2 \not\rightarrow (Q_1, c_4)$ and so $N(x_2, Q_1) \subseteq \{c_2, c_4, c_r\}$ for some $r \in \{1, 3\}$. Thus $x_2 \rightarrow (Q_1, c_{r+2})$. Hence $x_0 \not\Rightarrow (Q_1, c_{r+2})$. This implies that $c_1c_3 \in E$. Then $e(x_2, Q_1) = 2$ with $x_2c_4 \in E$ as $x_2 \not\rightarrow (Q_1, c_4)$. Hence $i(x_0x_3, Q_1) \neq 0$ and so $x_2 \rightarrow (Q_1; x_0x_1x_3)$, a contradiction. Therefore $e(T, Q_i) \leq 10$ for all Q_i in H_1. As $G_1 \not\supseteq 2C_4$, $e(c_i, T) \leq 1$ for all $c_i \in V(Q_1)$. Thus $e(F, G_1) \leq 15$. Let $c_r \in N(x_2x_3, Q_1)$. If $e(c_r, G_1) \leq 4$ then $e(F + c_r, G_1) \leq 19$ and so $e(F + c_r, H_1) \geq 10k - 19 = 10(k-2) + 1$. Thus $e(F + c_r, Q_i) \geq 11$ for some Q_i in H_1. As $e(T, Q_i) \leq 10$ and by Lemma 4.14, $x_0 \not\Rightarrow (Q_1, c_r)$. Therefore for each $c_r \in N(x_2x_3, Q_1)$, either $e(c_r, G_1) = 5$ or $x_0 \not\Rightarrow (Q_1, c_r)$. Hence $c_4 \notin N(x_2x_3, Q_1)$ and so $e(x_2x_3, c_1c_2c_3) = 3$. Then $c_1c_3 \in E$ for otherwise $e(c_1, G_1) = 4$ and $x_0 \Rightarrow (Q_1, c_1)$. Since $x_2 \not\rightarrow (Q_1; x_0x_1x_3)$, $e(x_3, Q_1) = 0$. ∎

Proof of Claim 2.6. Suppose that the claim is false. W.l.o.g., say $Q_1 = c_1c_2c_3c_4c_1$, $e(x_0, Q_1) = 4$ and $e(x_2x_3, Q_1) \geq 1$. By Lemma 4.1(a), $\tau(Q_1) = 2$. By Lemma 4.16, $e(x_1, Q_1) = 0$, $e(x_r, Q_1) \leq 1$ for $r \in \{2, 3\}$ and $e(T, Q_i) \geq 11$ for some Q_i in H_1. W.l.o.g., say $x_2c_4 \in E$ and $e(T, Q_2) \geq 11$. By Lemma 4.1(b), $\tau(Q_2) = 2$. Among all the strong feasible chains σ with these properties, we may assume that σ is chosen such that $e(Q_2 + x_3, Q_1)$ is maximal.

Let $T_1 = \{c_1, c_2, c_3\}$ and $G_0 = [T, Q_2]$. Let $i \in \{3, \ldots, k-1\}$. Note that $G_0 + y \supseteq 2C_4$ for all $y \in V(G) - V(G_0)$ with $e(y, G_0) \geq 2$. As $[G_2, Q_i] \not\supseteq 4C_4$, this implies that $x \not\rightarrow (Q_i; V(G_0))$ and $e(x, G_0) \leq 1$ for all $x \in T_1 \cup \{x_0, c_4\}$. Moreover, for each $U \subseteq V(G_0)$ with $|U| = 3$, $G_0 - U \supseteq C_4$ and so $[x, U, Q_i] \not\supseteq 2C_4$ for all $x \in T_1 \cup \{x_0, c_4\}$. As $[x_0, c_4, x_2, x_1] \supseteq C_4$ and $G_0 - \{x_1, x_2, u\} \supseteq C_4$ for all $u \in V(G_0 - \{x_1, x_2\})$, $[u, Q_i, T_1] \not\supseteq 2C_4$ for all $u \in V(G_0 - \{x_1, x_2\})$. This implies that $u \not\rightarrow (Q_i; T_1)$ and $e(u, T_1) \leq 1$ for all $u \in V(G_0 - \{x_1, x_2\})$. Since $G_0 - \{u, v, x_i\} \supseteq C_4$ for each

$\{u,v\} \subseteq V(G_0 - \{x_1, x_2\})$ with $u \neq v$ and $i \in \{1,2\}$, it follows that if $vx_1 \in E$ then $u \not\rightarrow (Q_i; vx_1x_0)$ and if $vx_2 \in E$ then $u \not\rightarrow (Q_i; c_4x_2v)$. These properties will be used several times in the following argument. We claim that for each Q_i in H_2 with $e(G_2 - \{x_1, x_2\}, Q_i) \geq 21$, one of (40) and (41) holds:

$$e(x_0c_4, Q_i) = 0, e(c_r, Q_i) \leq 1 \text{ for all } c_r \in T_1, e(d, T_1) \leq 1 \text{ for all } d \in V(Q_i); \tag{40}$$
$$e(u, Q_i) \leq 1 \text{ for all } u \in V(G_0) - \{x_1, x_2\}, e(d, G_0 - \{x_1, x_2\}) \leq 1 \text{ for all } d \in V(Q_i) \tag{41}$$

Proof. Note that $\sigma' = (c_4x_2, T, Q_1', Q_2 \ldots, Q_{k-1})$ is a strong feasible chain and so x_0 and c_4 are in the symmetric position in our argument where $Q_1' = x_0c_1c_2c_3x_0$. Set $Z_1 = V(Q_1 + x_0)$ and $Z_2 = G_0 - \{x_1, x_2\}$. Suppose that $e(x, Q_i) \geq 3$ for some $x \in Z_1$. Then $x \rightarrow Q_i$ by Lemma 4.1(a). Thus for each $d \in V(Q_i)$, $e(d, Z_2) \leq 1$ as $x \not\rightarrow (Q_i; V(G_0))$. Hence $e(Z_2, Q_i) \leq 4$ and so $e(Z_1, Q_i) \geq 17$. Thus $e(x, Q_i) = 4$ for some $x \in Z_1$ and $e(T_1, Q_i) \geq 9$. By Lemma 4.1(a), $\tau(Q_i) = 2$. If $e(u, Q_i) \geq 2$ for some $u \in Z_2$ then $u \rightarrow (Q_i, d)$ for some $d \in V(Q_i)$ with $e(d, T_1) \geq 2$. Thus $u \rightarrow (Q_i; T_1)$, a contradiction. Hence $e(u, Q_i) \leq 1$ for all $u \in Z_2$. Thus (41) holds.

Therefore we may assume that $e(x, Q_i) \leq 2$ for all $x \in Z_1$. Thus $e(Q_i, Z_1) \leq 10$ and so $e(Z_2, Q_i) \geq 11$. Suppose that $e(y, Q_i) = 2$ for some $y \in Z_1$. Assume for the moment $e(y, dd^*) = 2$ for some $d \in V(Q_i)$. Say $Q_i = d_1d_2d_3d_4d_1$ with $e(y, d_1d_3) = 2$. Then $e(d_2, G_0) \leq 1$ and $e(d_4, G_0) \leq 1$ as $y \not\rightarrow (Q_i; V(G_0))$. Thus $e(d_1d_3, Z_2) \geq 9$ and $e(Z_2, Q_i) \leq 12$. Clearly, $e(Z_1, Q_i) \geq 21 - 12 = 9$ and so $e(T_1, Q_i) \geq 5$. Thus for each $u \in Z_2$, $u \not\rightarrow Q_i$ as $u \not\rightarrow (Q_i; T_1)$ and so $e(u, Q_i) \leq 3$. As $e(Z_2, Q_i) \geq 11$, $e(z_1, Q_i) = 3$ for some $z_1 \in Z_2$. Suppose that $e(z_1, d_2d_4) = 2$. Then w.l.o.g., say $z_1d_1 \in E$. Then $d_1d_3 \notin E$ as $z_1 \not\rightarrow Q_i$. As $e(d_1d_3, Z_2) \geq 9$, $e(d_l, Z_2) = 5$ for some $l \in \{1,3\}$. By Lemma 3.1(b), $G_0 + d_l \supseteq 2K_4$. Say $\{l, m\} = \{1,3\}$. As $z_1 \not\rightarrow (Q_i; T_1)$, $e(d_j, T_1) \leq 1$ for $j \in \{1,3\}$. For each $c_r \in T_1$, $e(c_r, d_2d_4) \leq 1$ as $c_r \not\rightarrow (Q_i; V(G_0))$. As $e(T_1, Q_i) \geq 5$, it follows that $e(c_r, d_md_t) = 2$ for some $c_r \in T_1$ and $t \in \{2,4\}$. Thus $[c_r, d_m, d_t] \cong C_3$ and so $[G_2, Q_i] \supseteq C_3 \uplus 3K_4$. By (1), $\tau(Q_i) = 2$, a contradiction. Hence $e(z_1, d_2d_4) = 1$. W.l.o.g., say $e(z_1, d_1d_2d_3) = 3$. Then $d_2d_4 \notin E$ as $z_1 \not\rightarrow Q_i$. Clearly, $e(z_2, d_1d_3) = 2$ for some $z_2 \in Z_2 - \{z_1\}$ as $e(d_1d_3, Q_i) \geq 9$. Since $e(T_1, Q_i) \leq 6$, $e(x_0c_4, Q_i) \geq 9 - 6 = 3$. W.l.o.g., say $e(x_0, Q_i) = 2$. As $x_0 \not\rightarrow (Q_i; V(G_0))$, $e(x_0, d_2d_4) \leq 1$. Thus $e(x_0, d_1d_3) \geq 1$. Say w.l.o.g. $x_0d_1 \in E$. If $e(x_0, d_1d_3) = 1$, we have $x_0 \Rightarrow (Q_1, y)$ and $e(y, d_1d_3) = 2$. As $z_1 \not\rightarrow (Q_i; T_1)$, $e(d_r, T_1) \leq 1$ and so $e(d_r, Q_1) \leq 2$ for $r \in \{2,4\}$. Thus we obtain a contradiction with Lemma 4.9.

The above argument shows that no vertex of Z_1 is adjacent to two non-consecutive vertices of Q_i. It follows that $\tau(Q_i) \leq 1$ for otherwise we may choose a 4-cycle Q_i' from $[Q_i]$ such that y is adjacent to two non-consecutive vertices of Q_i' and obtain a contradiction in the above argument with Q_i' in place of Q_i. Hence $[G_2, Q_i] \not\supseteq C_3 \uplus 3K_4$

by (1). W.l.o.g., say $e(y, d_1d_2) = 2$. We claim that $\tau(Q_i) = 1$. As $e(d_1d_3, Z_2) + e(d_2d_4, Z_2) \geq 11$, say w.l.o.g. $e(d_1d_3, Q_i) \geq 6$. If $G_0 + d_1 \supseteq 2K_4$ then $[G_2, Q_i] \supseteq 2P_2 \uplus 3K_4$ since $[y, d_2, d_3, d_4] \supseteq P_4$. By Lemma 4.2, $\tau(Q_i) \geq 1$. Hence assume that $G_0 + d_1 \not\supseteq 2K_4$. By Lemma 3.1($b$), $e(d_1, Z_2) \leq 4$. By Lemma 3.1(d), if $e(d_1, Z_2) = 4$ then $e(x_1x_2, Q_2) = 7$ and so $e(x_3, Q_2) = 4$. It follows that $e(d_3, Z_2) \geq 6 - e(d_1, Z_2) \geq 2$ and $uv \in E$ for some $\{u, v\} \subseteq N(d_3, Z_2)$. By Lemma 3.1($b$), $[Z_2]$ has a triangle T' such that $uv \in E(T')$ and $G_0 - V(T') \cong K_4$. Thus $[T' + d_3] \supseteq C_4^+$. As $[y, d_1, d_2] \cong C_3$, we obtain that $[G_2, Q_i] \supseteq C_3 \uplus 2K_4 \uplus C_4^+$. By (1), $\tau(Q_i) \geq 1$. W.l.o.g, say $d_1d_3 \in E$. Then $y \to (Q_i, d_4)$ and so $e(d_4, Z_2) \leq 1$. As $[y, d_1, d_2] \cong C_3$ and $[d_1, d_4, d_3] \cong C_3$, we obtain that $G_0 + d_j \not\supseteq 2K_4$ for $j \in \{2, 3\}$ since $[G_2, Q_i] \not\supseteq C_3 \uplus 3K_4$. By Lemma 3.1($b$), $e(d_j, Z_2) \leq 4$ for $j \in \{2, 3\}$. Hence $e(Z_2, Q_i) \leq 14$. As $e(Z_2, Q_i) \geq 11$, $e(u, Q_i) \geq 3$ for some $u \in Z_2$. Then $u \to (Q_i, d_j)$ for $j \in \{2, 4\}$. If $e(x, d_1d_3d_4) = 2$ for some $x \in Z_1$ then $x \to (Q_i, d_2)$ and so $e(d_2, Z_2) \leq 1$. It follows that $e(Z_2, Q_i) = 11$ and $e(Z_1, Q_i) = 10$. Thus $e(T_1, Q_i) = 6$. As $e(c_r, d_1d_3) \leq 1$ for each $c_r \in T_1$, $e(d_2d_4, T_1) \geq 3$. Hence $e(d_j, T_1) \geq 2$ for some $j \in \{2, 4\}$ and so $u \to (Q_i; T_1)$, a contradiction. Therefore $e(x, d_1d_4d_3) \leq 1$ for all $x \in Z_1$. Thus $xd_2 \in E$ for each $x \in Z_1$ with $e(x, Q_i) = 2$. This implies that T_1 has at most one vertex c_r with $e(c_r, Q_i) = 2$ since $u \not\to (Q_i; T_1)$. Thus $e(T_1, Q_i) \leq 4$ and so $e(Z_1, Q_i) \leq 8$. Then $e(Z_2, Q_i) \geq 13$ and so $e(d_2, Z_2) \geq 13 - e(d_1d_3d_4, Z_2) \geq 3$. As $e(Z_2, Q_i) \leq 14$, $e(x_0c_4, Q_i) \geq 7 - e(T_1, Q_i) \geq 3$. Say w.l.o.g. $e(x_0, Q_i) = 2$. Then $x_0d_2 \in E$. As $e(d_2, Z_2) \geq 3$, $vx_1 \in E$ for some $v \in N(d_2, Z_2) - \{u\}$. Thus $u \to (Q_i, d_2; x_0x_1v)$, a contradiction.

Therefore $e(x, Q_i) \leq 1$ for all $x \in Z_1$. Thus $e(Z_2, Q_i) \geq 16$. We need show that $e(x_0c_4, Q_i) = 0$. On the contrary, say w.l.o.g. $x_0d_1 \in E$. Assume that $vd_1 \in E$ for some $v \in N(x_1, Z_2)$. Then for each $u \in Z_2 - \{v\}$, $u \not\to (Q_i, d_1)$ and so $e(u, d_2d_4) \leq 1$. As $e(Z_2, Q_i) \geq 16$, it follows that $e(v, Q_i) = 4$ and $e(u, Q_i) = 3$ with $ud_1 \in E$ for all $u \in Z_2 - \{v\}$. Thus $v \to (Q_i, d_1; x_0x_1u)$ for some $u \in N(x_1, Z_2) - \{v\}$, a contradiction. Hence $vd_1 \notin E$ for each $v \in N(x_1, Z_2)$. As $e(x_1, Z_2) \geq 4$ and $e(Z_2, Q_i) \geq 16$, it follows that $e(x_1, Z_2) = 4$ and $e(Z_2, Q_i) = 16$. Thus $e(Z_1, Q_i) = 5$ and so $e(c_4, Q_i) = 1$. Similarly, we shall have that $e(x_2, Z_2) = 4$. Thus $e(T, Q_2) \leq 10$, a contradiction. Hence $e(x_0c_4, Q_i) = 0$ and so $e(Z_2, Q_i) \geq 18$. Thus $e(u, Q_i) = 4$ for some $u \in Z_2$. As $u \not\to (Q_i; T_1)$, $e(d, T_1) \leq 1$ for all $d \in V(Q_i)$. Hence (40) holds. □

Let $N = [\cup Q_i]$ where i runs over $\{3, \ldots, k-1\}$ with $e(G_2 - \{x_1, x_2\}, Q_i) \geq 21$. We say that a vertex z is *attached* to a subgraph G' of G if $z \notin V(G')$ and $e(z, G') = 1$. We have the following four properties.

Property 1. If $xy \in E(T_1, Z_2)$, neither x nor y is attached to some Q_i in N.

To see this, say w.l.o.g. $xy = c_1u_1$ with $u_1 \in Z_2$ such that for some $v \in \{c_1, u_1\}$, v is attached to some Q_i in N. Assume $v = c_1$. By (40), $e(Z_2, Q_i) \geq 21 - e(T_1, Q_i) \geq 18$.

Let T' be a triangle of $[Z_2]$ with $u_1 \in V(T')$. As $e(Z_2, Q_i) \geq 18$, $e(T', Q_i) \geq 10$. By Lemma 3.4(a), $[c_1, T', Q_i] \supseteq 2C_4$, a contradiction. Hence $v = u_1$. By (41), $e(Z_2, Q_i) \leq 4$ and $e(Z_1, Q_i) \geq 17$. Then $e(T_1, Q_i) \geq 9$. As $e(x, Q_i) = 4$ for some $x \in Z_1$, we have $\tau(Q_i) = 2$ by Lemma 4.1(a). By Lemma 3.4(b), $[u_1, T_1, Q_i] \supseteq 2C_4$, a contradiction. □

Property 2. For each Q_i in N, if (41) holds for Q_i then $\tau(Q_i) = 2$ and $e(T_1, Q_i) \geq 10$. Furthermore, if $e(T_1, Q_i) = 10$ then $e(Z_2, T_1) \geq 2$.

To see this, we have $e(Z_1, Q_i) \geq 21 - e(Z_2, Q_i) \geq 17$. Thus $e(x, Q_i) = 4$ for some $x \in Z_1$. By Lemma 4.1(a), $\tau(Q_i) = 2$. Clearly, if $e(Z_2, Q_i) \leq 2$ then $e(T_1, Q_i) \geq 21 - 8 - 2 = 11$. For the proof, we may assume that $e(Z_2, Q_i) \geq 3$ and $e(T_1, Q_i) \leq 10$. W.l.o.g., say $Q_i = Q_3$. Then $e(x_0c_4, Q_3) \geq 7$. W.l.o.g., say $e(x_0, Q_3) = 4$ and $e(c_4, Q_3) \geq 3$. As $e(Z_1, Q_3) \geq 17$, $e(d, Z_1) = 5$ for some $d \in V(Q_3)$. Assume $e(c_4, Q_3) = 4$. Then we replace c_4 with d in Q_1 and replace d with c_4 in Q_3 to obtain two disjoint 4-cycles C' and C'', respectively. Clearly, $\tau(C') = \tau(C'') = 2$. Thus $(x_0x_1, T, C'', Q_2, C', Q_4, \ldots, Q_{k-1})$ is a strong feasible chain such that $e(x_0, C'') = 4$, $e(x_2, C'') = 1$ and $e(Z_2, C'') \geq e(Z_2, Q_3) - 1$. By our assumption on σ, $e(Z_2, Q_1) \geq e(Z_2, Q_3) - 1$. By *Property* 1, $N(T_1, Z_2) \cap N(Q_3, Z_2) = \emptyset$. As $e(c_4, Z_2) = 0$, $E(Z_2, T_1) = E(Z_2, Q_1)$. It follows that $e(Z_2, Q_3) + e(Z_2, Q_3) - 1 \leq |Z_2|$. This yields that $e(Z_2, Q_3) \leq 3$. It follows that $e(Z_2, Q_3) = 3$, $e(T_1, Q_3) = 10$ and $e(Z_2, T_1) \geq 2$. Hence we may assume that $e(c_4, Q_3) = 3$. Then $e(Z_2, Q_3) = 4$ and $e(T_1, Q_3) = 10$. As $e(T, Q_2) \geq 11$, it is easy to see that $e(y, x_1x_2) = 2$ for some $y \in N(Q_3, Z_2)$ such that $G_0 - \{x_1, x_2, y\} \cong K_4$. Let $G_0 - \{x_1, x_2, y\} \supseteq Q' \cong C_4$. Then $(x_0x_1, x_1yx_2x_1, Q_1, Q', Q_3, \ldots, Q_{k-1})$ is a strong feasible chain with $e(x_0, Q_3) = 4$ and $e(y, Q_3) = 1$. Clearly, $e(x_1x_2y, Q') \geq 11$ and $e(G_0 - \{x_1, y\}, Q_3) = 3$. By the assumption on σ again, $e(Z_2, T_1) = e(Z_2, Q_1) = 3$. Thus $N(T_1, Z_2) \cap N(Q_3, Z_2) \neq \emptyset$, contradicting *Property* 1. □

Property 3. For each $v \in T_1 \cup Z_2$, v is attached to at most one Q_i in N.

To see this, suppose that for some $v \in T_1 \cup Z_2$, v is attached to some Q_j and Q_r in N with $j \neq r$. W.l.o.g., say $Q_j = Q_3$ and $Q_r = Q_4$. Say $e(v, u_1w_1) = 2$ where $Q_3 = u_1u_2u_3u_4u_1$ and $Q_4 = w_1w_2w_3w_4w_1$. First, suppose that $v \in T_1$. By (40), $e(Z_2, Q_3) \geq 18$ and $e(Z_2, Q_4) \geq 18$. Then $e(x, u_1w_1) = 2$, $e(y, Q_3) = 4$ and $e(z, Q_4) = 4$ for some $\{x, y, z\} \subseteq Z_2$ with $|\{x, y, z\}| = 3$. Thus $[v, u_1, x, w_1] \supseteq C_4$, $y \to (Q_3, u_1)$, $z \to (Q_4, w_1)$, $G_0 - \{x, y, z\} \supseteq C_4$ and $x_0 \to (Q_1, v)$, i.e., $G_4 \supseteq 5C_4$, a contradiction. Hence $v \in Z_2$. As $[x_0, c_4, x_2, x_1] \supseteq C_4$ and $G_0 - \{x_1, x_2, v\} \supseteq C_4$, we shall have that $[v, T_1, Q_3, Q_4] \not\supseteq 3C_4$. By (41) and *Property* 2, $\tau(Q_i) = 2$ and $e(T_1, Q_i) \geq 10$ for $i \in \{3, 4\}$. Suppose that $e(x, u_1w_1) = 2$ for some $x \in T_1$. As $e(T_1, Q_3) \geq 10$, $y \to (Q_3, u_1)$ for some $y \in T_1 - \{x\}$. Say $T_1 = \{x, y, z\}$. Then $z \not\to (Q_4, w_1)$. As $e(T_1, Q_4) \geq 10$, this implies that $e(z, Q_4) = 2$, $zw_1 \in E$ and $e(xy, Q_4) = 8$. If $zu_1 \in E$ then $[z, u_1, v, w_1] \supseteq C_4$, $x \to (Q_4, w_1)$ and $y \to (Q_3, u_1)$, a contradiction. Hence $zu_1 \notin E$. As $e(T_1, Q_3) \geq 10$, $e(z, Q_3) \geq 2$. Thus $z \to (Q_3, u_1)$ and $y \to (Q_4, w_1)$, a contradiction.

Therefore we may assume that for all $u \in V(Q_3)$, $w \in V(Q_4)$ and $v \in Z_2$ if $e(v, uw) = 2$ then $e(x, uw) \leq 1$ for all $x \in T_1$. Then $e(T_1, Q_i) \not\geq 11$ for some $i \in \{3, 4\}$. Say w.l.o.g. $e(T_1, Q_3) = 10$. Then $e(Z_2, Q_3) \geq 21 - 18 = 3$. By *Property* 2, $e(Z_2, T_1) \geq 2$. By *Property* 1, $N(T_1, Z_2) \cap N(Q_3, Z_2) = \emptyset$. It follows that $e(Z_2, Q_3) = 3$ and $e(Z_2, T_1) = 2$. Say $N(Q_3, Z_2) = \{v_1, v_2, v_3\}$. By *Property* 1, $N(Q_4, Z_2) \subseteq \{v_1, v_2, v_3\}$. Let $x \in T_1$ be such that $e(x, Q_3) = 4$. Then for any $w_i \in V(Q_4)$ with $e(w_i, \{v_1, v_2, v_3\}) = 1$, we shall have $xw_i \notin E$. It follows that $e(T_1, Q_4) \leq 12 - e(Z_2, Q_4)$ and consequently, $e(Z_1 \cup Z_2, Q_4) \leq 12 + e(x_0c_4, Q_4) \leq 20$, a contradiction. □

Let q be the number of vertices of $T_1 \cup Z_2$ that are attached to some Q_i in N. By *Property* 3, $e(Z_1 \cup Z_2, N) \leq q + 20p$ where $|V(N)| = 4p$. Let $r = e(T_1, Z_2)$. By *Property* 1, $q \leq 8 - 2r$. Clearly, $e(Z_1 \cup Z_2, G_2) \leq 52 + 2r$ and if the equality holds then $e(Z_2, G_0) = 30$, i.e., $G_0 \cong K_7$. Then $e(Z_1 \cup Z_2, H_2) \geq 20k - 52 - 2r = 20(k-3) + 8 - 2r$. As $e(Z_1 \cup Z_2, Q_i) \leq 20$ for all Q_i in $H_2 - V(N)$, we obtain that $e(Z_1 \cup Z_2, N) \geq 20p + 8 - 2r$. This yields that $q = 8 - 2r$ and $e(Z_1 \cup Z_2, N) = 20p + 8 - 2r$. It follows that $G_0 \cong K_7$, $e(Z_1, Q_i) = 20$ for all Q_i in N for which (41) holds and $e(Z_2, Q_i) = 20$ for all Q_i in N for which (40) holds. We claim that $r = 3$. If not, let $v \in Z_2$ be attached to some Q_i in N with $e(Z_1, Q_i) = 20$ and $\tau(Q_i) = 2$. Say $Q_i = Q_3$ and $vd \in E$ with $d \in V(Q_3)$. Let $c_r \in T_1$ be such that $e(c_r, Z_2) = 0$. Then we replace c_r with d in Q_1 and replace d with c_r in Q_3 to obtain two disjoint 4-cycles C' and C'' such that $\tau(C') = \tau(C'') = 2$, $e(Z_2, C') = r + 1$, $e(x_0, C') = 4$ and $e(x_2, C') = 1$. By the assumption on σ, $e(Z_2, Q_1) \geq r + 1$, i.e., $e(T_1, Z_2) \geq r + 1$, a contradiction.

Say $E(T_1, Z_2) = \{c_1u_1, c_2u_2, c_3u_3\}$ and let $T_2 = \{u_1, u_2, u_3\}$ and Q_2' a 4-cycle in $G_0 - T_2$. Clearly, $e(T_1 \cup T_2, G_2) = 36$ and so $e(T_1 \cup T_2, H_2) \geq 12k - 36 = 12(k-3)$.

Property 4. For each Q_i in H_2 with $e(T_1 \cup T_2, Q_i) \geq 12$, either $e(T_1, Q_i) = 0$, or $e(T_2, Q_i) = 0$, or $e(T_2, Q_i) = 6$ and $e(c_r, Q_i) = 2$ for all $c_r \in T_1$.

To see this, first assume that $e([T_2] + c_r, Q_i) \geq 9$ for some $c_r \in T_1$. Let $Q^{(r)}$ be a 4-cycle in $[Q_1 - c_r + x_0]$. Then $(c_ru_r, [T_2], Q^{(r)}, Q_2', Q_3, \ldots, Q_{k-1})$ is a strong feasible chain . By Claim 2.2, we see that $e(c_r, Q_i) = 0$ for otherwise $e([T_2] + c_r, Q_i) = 9$ and $[c_t, T_2, Q_i] \supseteq 2C_4$ where $c_t \in T_1$ with $e(c_t, Q_i) \geq 2$, a contradiction. Thus $e(T_2, Q_i) \geq 9$. Let r run over $\{1, 2, 3\}$, we see that $e(T_1, Q_i) = 0$. Hence we may assume that $e([T_2] + c_r, Q_i) \leq 8$ for all $r \in \{1, 2, 3\}$. If $e(c_r, Q_i) \leq 2$ for all $r \in \{1, 2, 3\}$ then the third statement of the property follows. Hence assume that $e(c_r, Q_i) \geq 3$ for some $c_r \in T_1$. By Lemma 4.1(a), $\tau(Q_i) \geq 1$ and $c_r \to Q_i$. As $c_r \not\to (Q_i; V(G_0))$, $e(d, T_2) \leq 1$ for all $d \in V(Q_i)$. Thus $e(T_1, Q_i) \geq 12 - e(T_2, Q_i) \geq 8$. Suppose that $e(T_1, Q_i) \geq 9$. If there exists $u_t \in T_2$ with $e(u_t, Q_i) \geq 1$, then $[T_1, u_t, Q_i] \supseteq 2C_4$ by

Lemma 3.4(b), a contradiction. Therefore $e(T_2, Q_i) = 0$. Hence we may assume that $e(T_1, Q_i) = 8$. Then $e(T_2, Q_i) = 4$ and $e(u_t, Q_i) \geq 2$ for some $u_t \in T_2$. Say w.l.o.g. $e(u_1, Q_i) \geq 2$. Suppose that $e(u_1, dd^*) = 2$ for some $d \in V(Q_i)$. Say $Q_i = d_1d_2d_3d_4d_1$ with $e(u_1, d_1d_3) = 2$. Then $e(d_j, T_1) \leq 1$ for $j \in \{2, 4\}$. It follows that $e(d_1d_3, T_1) = 6$, $e(d_j, T_1) = 1$ for $j \in \{2, 4\}$ and so $e(c_s, d_1d_2d_3) = 3$ for some $c_s \in T_1$. By Lemma 4.1(a), $d_2d_4 \in E$. As $u_1 \not\to (Q_i; T_1)$, $e(u_1, d_2d_4) = 0$. Thus $e(u_2u_3, d_2d_4) = 2$. As $u \not\to (Q_i; T_1)$ for each $u \in \{u_2, u_3\}$, we see that $e(u, d_2d_4) = 1$ for each $u \in \{u_2, u_3\}$. Thus $[u_2, u_3, d_2, d_4] \supseteq C_4$, $[c_1, d_1, u_1, d_3] \supseteq C_4$ and so $[c_1, Q_i, T_2] \supseteq 3C_4$, a contradiction. This argument shows that no vertex of T_2 is adjacent to two non-consecutive vertices of Q_i. This implies that $\tau(Q_i) \neq 2$ for otherwise we may choose a 4-cycle Q_i' from $[Q_i]$ such that u_1 is adjacent to two non-consecutive vertices of Q_i' and then obtain a contradiction as above. Say w.l.o.g. $e(u_1, d_1d_2) = 2$. As $\tau(Q_i) \geq 1$, say w.l.o.g. $d_1d_3 \in E$. Then $e(d_4, T_1) \leq 1$ as $u_1 \not\to (Q_i; T_1)$. Thus $e(T_1, d_1d_2d_3) \geq 7$ and so $e(c_t, d_1d_2d_3) = 3$ for some $c_t \in T_1$. By Lemma 4.1(a), $d_2d_4 \in E$ and so $\tau(Q_i) = 2$, a contradiction. □

By *Property* 4, $e(T_1 \cup T_2, Q_i) = 12$ for all Q_i in H_2. Let s_1 be the number of all the Q_i in H_2 with $e(T_1, Q_i) = 0$. Let s_2 be the number of all the Q_i in H_2 with $e(T_2, Q_i) = 0$. Let s_3 be the number of all the Q_i in H_1 with $e(T_2, Q_i) = 6$ and $e(c_r, Q_i) = 2$ for all $c_r \in T_1$. Then $s_1 + s_2 + s_3 = k - 3$ by *Property* 4. If $s_1 \geq s_2$, $e(c_1, G) = 5 + 4s_2 + 2s_3 \leq 5 + 2s_1 + 2s_2 + 2s_3 = 2k - 1$, a contradiction. Hence $s_1 < s_2$. Then $e(T_2, G) = 21 + 12s_1 + 6s_3 = 6k - 6(s_2 - s_1) + 3 \leq 6k - 3$, a contradiction. ∎

Proof of Claim 2.7. Suppose that the claim is false. By Lemma 4.16, we may assume that $Q_1 = c_1c_2c_3c_4c_1$ with $\tau(Q_1) = 2$, $N(x_0, Q_1) = N(x_2, Q_1) = \{c_1, c_2, c_3\}$ and $e(x_3, Q_1) = 0$. Moreover, $e(T, Q_i) \leq 10$ for all Q_i in H_1. We have the following property.

Property A. For any strong feasible chain $(y_0y_1, C, J_1, \ldots, J_{k-1})$ with $y_1 \in V(C)$, there exist two labellings $C = y_1y_2y_3y_1$ and $J_i = v_1v_2v_3v_4v_1$ for some $i \in \{1, \ldots, k-1\}$ such that $N(y_0, J_i) = N(y_2, J_i) = \{v_1, v_2, v_3\}$ and $\tau(J_i) = 2$. Moreover, $e(C, J_r) \leq 10$ for all $r \in \{1, \ldots, k-1\}$.

To see this, let $C = y_1y_2y_3y_1$ and $L = \cup_{i=1}^{k-1} J_i$. Then $2e(y_0, L) + e(y_2y_3, L) \geq 8k - 6 = 8(k-1) + 2$. Thus $2e(y_0, J_i) + e(y_2y_3, J_i) \geq 9$ for some $i \in \{1, \ldots, k-1\}$. If $e(y_0, J_i) \leq 2$ then $e(y_0y_2y_3, J_i) \geq 7$. By Claim 2.5, $e(y_0, J_i) \leq 1$ and if the equality holds then $e(y_0y_2y_3, J_i) = 7$. It follows that $2e(y_0, J_i) + e(y_2y_3, J_i) \leq 8$, a contradiction. Hence $e(y_0, J_i) \geq 3$. If $e(y_0, J_i) = 4$ then $e(y_2y_3, J_i) = 0$ by Claim 2.6 and so $2e(y_0, J_i) + e(y_2y_3, J_i) = 8$, a contradiction. Thus $e(y_0, J_i) = 3$ and $e(y_2y_3, J_i) \geq$

3. Then the property follows from Lemma 4.16. □

Clearly, $e(F+c_4, G_1) \leq 19$ and so $e(F+c_4, H_1) \geq 10(k-2)+1$. Say $e(F+c_4, Q_2) \geq 11$. We break into the following two cases.

Case 1. $e(F, Q_2) \geq 9$.

By Claim 2.2, we see that $e(x_0, Q_2) = 0$ for otherwise $e(F, Q_2) = 9$, $[T, Q_2, c_4] \supseteq 2C_4$ and so $G_2 \supseteq 3C_4$. As $x_0 \Rightarrow (Q_1, c_4)$ and by Lemma 3.2, $e(c_4, Q_2) \leq 1$. By *Property A*, $e(T, Q_2) = 10$ and $e(c_4, Q_2) = 1$. If $e(c_i, Q_2) \geq 1$ for some $i \in \{1, 2, 3\}$ then $e(T + c_i, Q_2) \geq 11$, and by Lemma 3.4(a), $[T + c_i, Q_2] \supseteq 2C_4$ and so $G_2 \supseteq 3C_4$, a contradiction. Hence $e(c_1c_2c_3, Q_2) = 0$. Let $w_1 \in V(Q_2)$ be such that $w_1c_4 \in E$ and $G_0 = [T, Q_2]$. We claim that there exists no triangle $T_1 = w_1u_2u_3w_1$ in $G_0 - \{x_1, x_2\}$ such that (42) holds and there exists no triangle $T_2 = x_1v_2v_3x_1$ in $G_0 - \{w_1, x_2\}$ such that (43) holds:

$$G_0 - V(T_1) \geq Q_2, x_1u_2 \in E, G_0 - \{u_2, u_3, x_1\} \supseteq C_4, G_0 - \{x_2, w_1, u_i\} \supseteq C_4 (i = 2, 3); \tag{42}$$
$$G_0 - V(T_2) \geq Q_2, w_1v_2 \in E, G_0 - \{v_2, v_3, w_1\} \supseteq C_4, G_0 - \{x_2, x_1, v_i\} \supseteq C_4 (i = 2, 3) \tag{43}$$

To see this, suppose that (42) holds first. Let $R = \{c_4, u_2, u_3, x_0\}$ and $F_1 = c_4w_1u_2u_3w_1$. Then $e(R, G_2) \leq 23$ and so $e(R, H_2) \geq 8(k-3)+1$. Say $e(R, Q_3) \geq 9$. Assume $e(x_0, Q_3) \geq 3$. Then $x_0 \rightarrow Q_3$ and so $x_0 \rightarrow (Q_3; xw_1y)$ for some $\{x, y\} \in \{c_4, u_2, u_3\}$ with $x \neq y$. If $\{x, y\} = \{u_2, u_3\}$ then $[x_0, T_1, Q_3] \supseteq 2C_4$ and so $G_3 \supseteq 4C_4$ as $G_0 - V(T_1) \supseteq C_4$. Hence $c_4 \in \{x, y\}$. Say $x = c_4$ and $y = u_i$ with $i \in \{2, 3\}$. Then $[x_0, c_4, w_1, u_i, Q_3] \supseteq 2C_4$ and $x_2 \rightarrow (Q_1, c_4)$. Thus $G_3 \supseteq 4C_4$ as $G_0 - \{x_2, w_1, u_i\} \supseteq C_4$, a contradiction. Therefore $e(x_0, Q_3) \leq 2$ and so $e(u_2u_3c_4, Q_3) \geq 7$. Let $Q' = x_0c_1c_2c_3x_0$ and Q'' a 4-cycle of $G_0 - V(T_1)$. Then $\sigma_1 = (c_4w_1, T_1, Q', Q'', Q_3, \ldots, Q_{k-1})$ is a strong feasible chain. By Claim 2.5, either $e(c_4, Q_3) = 0$ or (9) holds w.r.t. F_1 and Q_3. By Lemma 4.2, $[F_1, Q_3] \not\supseteq P \uplus Q$ with $P \supseteq 2P_2$, $Q \cong C_4$ and $\tau(Q) = \tau(Q_3) + 2$. By Lemma 3.3, there exists a labelling $Q_3 = d_1d_2d_3d_4d_1$ such that $e(u_2u_3, d_2d_3d_4) = 6$ and $x_0d_3 \in E$. Thus $u_3 \rightarrow (Q_3; x_0x_1u_2)$ and so $G_3 \supseteq 4C_4$ since $G_0 - \{u_2, u_3, x_1\} \supseteq C_4$, a contradiction. Hence (42) does not hold with T_1. Similarly, (43) does not hold by an analogous argument with x_0 and T_2 in place of c_4 and T_1.

In order to find the above mentioned T_1 or T_2 in G_0, we claim that there exists a 4-cycle $a_1a_2a_3a_4a_1$ in $[Q_2]$ such that one of (44) to (55) holds below. To see them, we have $\tau(Q_2) \geq 1$ by Lemma 4.1(b). Moreover if $\tau(Q_2) = 1$ then for some $x_i \in V(T)$, $N(x_i, Q_2) = \{a, a^*\}$ for some $a \in V(Q_2)$ with $aa^* \in E$. Thus if $\tau(Q_2) = 1$ then one of (45), (51) and (53) holds. If $e(x_i, Q_2) = 2$ for some $x_i \in V(T)$ and $\tau(Q_2) = 2$ then one of (44), (50) and (52) holds. If $e(x_i, Q_2) = 4$, $e(x_j, Q_2) = e(x_l, Q_2) = 3$ for some permutation (i, j, l) of $\{1, 2, 3\}$ then one of (46), (48) and (54) holds if $N(x_j, Q_2) = N(x_l, Q_2)$. Otherwise one of (47), (49) and (55) holds.

$$e(x_2x_3, Q_2) = 8, e(x_1, a_1a_3) = 2, a_1a_3 \in E, a_2a_4 \in E; \tag{44}$$

$$e(x_2x_3, Q_2) = 8, e(x_1, a_1a_3) = 2, a_1a_3 \in E, a_2a_4 \notin E; \tag{45}$$
$$e(x_3, Q_2) = 4, e(x_1x_2, a_1a_2a_3) = 6, a_1a_3 \in E, a_2a_4 \in E; \tag{46}$$
$$e(x_3, Q_2) = 4, e(x_1, a_1a_2a_4) = 3, e(x_2, a_1a_2a_3) = 3, a_1a_3 \in E, a_2a_4 \in E; \tag{47}$$
$$e(x_2, Q_2) = 4, e(x_1x_3, a_1a_2a_3) = 6, a_1a_3 \in E, a_2a_4 \in E; \tag{48}$$
$$e(x_2, Q_2) = 4, e(x_1, a_1a_2a_4) = 3, e(x_3, a_1a_2a_3) = 3, a_1a_3 \in E, a_2a_4 \in E; \tag{49}$$
$$e(x_1x_3, Q_2) = 8, e(x_2, a_1a_2) = 2, a_1a_3 \in E, a_2a_4 \in E; \tag{50}$$
$$e(x_1x_3, Q_2) = 8, e(x_2, a_1a_3) = 2, a_1a_3 \in E, a_2a_4 \notin E; \tag{51}$$
$$e(x_1x_2, Q_2) = 8, e(x_3, a_1a_2) = 2, a_1a_3 \in E, a_2a_4 \in E; \tag{52}$$
$$e(x_1x_2, Q_2) = 8, e(x_3, a_1a_3) = 2, a_1a_3 \in E, a_2a_4 \notin E; \tag{53}$$
$$e(x_1, Q_2) = 4, e(x_2x_3, a_1a_2a_3) = 6, a_1a_3 \in E, a_2a_4 \in E; \tag{54}$$
$$e(x_1, Q_2) = 4, e(x_2, a_1a_2a_4) = 3, e(x_3, a_1a_2a_3) = 3, a_1a_3 \in E, a_2a_4 \in E. \tag{55}$$

We now claim that (55) and each of (46) to (52) do not hold. First, If (46) holds, we may assume w.l.o.g. $w_1 \in \{a_1, a_4\}$ and let $\{w_1, u_2, u_3\} = \{a_1, a_3, a_4\}$ with $u_2 = a_3$. If (47) holds, we may assume $w_1 \in \{a_1, a_3, a_4\}$ and let $\{w_1, u_2, u_3\} = \{a_1, a_3, a_4\}$ with $u_2 \in \{a_1, a_4\}$. If (48) holds then we may assume $w_1 \in \{a_1, a_4\}$. Furthermore, if $w_1 = a_4$, let $u_2 = a_1$ and $u_3 = a_3$ and if $w_1 = a_1$, let $v_2 = x_3$ and $v_3 = a_2$. If (49) holds, we may assume $w_1 \in \{a_1, a_3, a_4\}$ and let $\{w_1, u_2, u_3\} = \{a_1, a_3, a_4\}$ with $u_2 \in \{a_1, a_4\}$. If (50) holds, we may assume $w_1 \in \{a_1, a_4\}$ and let $\{w_1, u_2, u_3\} = \{a_1, a_3, a_4\}$ with $u_2 \in \{a_1, a_4\}$. If (51) holds, we may assume $w_1 \in \{a_1, a_4\}$. Furthermore, if $w_1 = a_4$, let $v_2 = x_3$ and $v_3 = a_2$ and if $w_1 = a_1$, let $v_2 = a_2$ and $v_3 = a_3$. If (52) holds, we may assume $w_1 \in \{a_1, a_4\}$. Furthermore, if $w_1 = a_1$, let $v_2 = a_4$ and $v_3 = a_3$ and if $w_1 = a_4$, let $u_2 = a_3$ and $u_3 = a_2$. If (55) holds, we may assume $w_1 \in \{a_1, a_3, a_4\}$ and let $\{w_1, u_2, u_3\} = \{a_1, a_3, a_4\}$ with $u_2 \in \{a_1, a_3\}$. Then (42) holds with $T_1 = w_1u_2u_3w_1$ and (43) holds with $T_2 = x_1v_2v_3x_1$, a contradiction.

Therefore one of (44), (45), (53) and (54) holds. If (44) or (45) holds, we may assume $w_1 \in \{a_1, a_2\}$. If (53) or (54) holds, we may assume $w_1 \in \{a_1, a_4\}$. If (44) holds with $w_1 = a_2$, let $u_2 = a_1$ and $u_3 = a_4$. If (45) holds with $w_1 = a_2$, let $u_2 = a_1$ and $u_3 = a_3$. With $T_1 = w_1u_2u_3w_1$, (42) holds, a contradiction. Hence if (44) or (45) holds, then $w_1 = a_1$. Let T' be a triangle of G_0 such that if (44) or (45) holds then $V(T') = \{x_1, a_1, a_3\}$ and if (53) or (54) holds then $V(T') = \{x_1, a_1, a_4\}$. Then $G_0 - V(T') \geq Q_2$. Let C be a 4-cycle in $G_0 - V(T')$ and $F' = T' + x_0x_1$. Then $\sigma_2 = (x_0x_1, T', Q_1, C, Q_3, \ldots, Q_{k-1})$ is a strong feasible chain with $x_0 \Rightarrow (Q_1, c_4)$ and $e(c_4, T') = 1$. Since $e(c_i, Q_2) = 0$ for $i \in \{1, 2, 3\}$, it follows that $e(F' + c_4, G_2) \leq 27$ and so $e(F' + c_4, H_2) \geq 10(k-3) + 3$. Thus $e(F' + c_4, Q_i) \geq 11$ for some Q_i with $3 \leq i \leq k-1$. By Lemma 4.14, $e(T', Q_3) \geq 11$, contradicting *Property A*.

Case 2. $e(F, Q_2) \leq 8$.

In this case, $e(c_4, Q_2) \geq 3$ and so $c_4 \to Q_2$. Thus $e(u, T) \leq 1$ for all $u \in V(Q_2)$ and so $e(x_0c_4, Q_2) \geq 7$. By Lemma 4.1(a), $\tau(Q_2) = 2$. Assume $e(x_2x_3, Q_2) \geq 1$. Then $e(x_0, Q_2) \neq 4$ by Claim 2.6. It follows that $e(c_4, Q_2) = 4$, $e(x_0, Q_2) = 3$ and $e(T, Q_2) = 4$. As $c_4 \not\to (Q_2; x_0x_1x_3)$, $i(x_0x_3, Q_2) = 0$ and so $e(x_3, Q_2) \leq 1$. If $e(x_3, Q_2) = 1$, say $Q_2 = b_1b_2b_3b_4b_1$ with $x_3b_4 \in E$ and $e(x_0, b_1b_2b_3) = 3$. By Lemma 4.16, $e(x_2x_3, Q_2) \leq 2$. Thus $e(x_2, Q_2) \leq 1$ and so $e(x_1, Q_2) \geq 2$. W.l.o.g., say $x_1b_1 \in E$. Then $[x_1, x_3, b_1, b_4] \supseteq C_4$, $[x_0, b_2, c_4, b_3] \supseteq C_4$ and so $G_2 \supseteq 3C_4$, a contradiction. Hence $e(x_3, Q_2) = 0$. Thus $e(x_2, Q_2) \geq 1$. By Claim 2.4, $e(x_0x_2, Q_2) \leq 6$. Thus $e(x_2, Q_2) \leq 3$ and $e(x_1, Q_2) \geq 1$. Let $u \in V(Q_2)$ with $x_2u \in E$. Let $[Q_2 - u + c_4] \supseteq C \cong C_4$. Then $(ux_2, T, C, x_0c_1c_2c_3x_0, Q_3, \ldots, Q_{k-1})$ is a strong feasible chain with $e(u, C) = 4$ and $e(x_1, C) \geq 1$, contradicting Claim 2.6. Hence $e(x_2x_3, Q_2) = 0$ and so $e(x_0x_1c_4, Q_2) \geq 11$. Let $Q_2 = u_1u_2u_3u_4u_1$ be such that $u_1 \in I(x_0x_1, Q_2)$ and $e(c_4, Q_2 - u_1) = 3$. Let $Q' = x_2c_1c_2c_3x_2$ and $Q'' = c_4u_2u_3u_4c_4$. Then $\sigma_3 = (x_3x_1, x_1x_0u_1x_1, Q', Q'', Q_3, \ldots, Q_{k-1})$ is a strong feasible chain. By *Property A*, for some Q_i with $3 \leq i \leq k-1$, say $Q_i = Q_3 = d_1d_2d_3d_4d_1$, such that $\tau(Q_3) = 2$ and $N(x_3, Q_3) = N(z, Q_3) = \{d_1, d_2, d_3\}$ for some $z \in \{x_0, u_1\}$. Furthermore, by the above argument, for some Q_j with $4 \leq j \leq k-1$, say $Q_j = Q_4$, such that $\tau(Q_4) = 2$ and $e(x_3x_1d_4, Q_4) \geq 11$. Let $w \in I(x_1x_3, Q_4)$. Then $d_4 \to (Q_4, w; x_1x_2x_3)$ and $z \to (Q_3, d_4)$. As $x_0 \to (Q_2, z)$ if $z = u_1$, we obtain $[F, Q_2, Q_3, Q_4] \supseteq 4C_4$, a contradiction. ∎

Acknowledgement The author thanks the anonymous referee for his very careful reading and corrections.

5 References

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