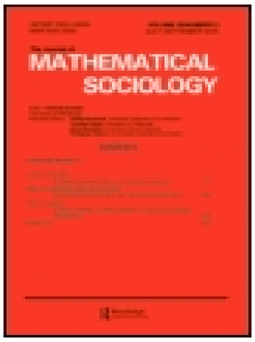

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Voter models and external influence

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ABSTRACT
In this paper, we extend the voter model (VM) and the threshold voter model (TVM) to include external influences modeled as a jump process. We study the newly-formulated models both analytically and computationally, employing diffusion approximations and mean field approximations. We derive results pertaining to the probability of reaching consensus on a particular opinion and also the expected consensus time. We find that although including an external influence leads to a faster consensus in general, this effect is more pronounced in the VM as compared to the TVM. Our findings suggest the potential importance of external influences in addition to local interactions.

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KEYWORDS
Voter model; threshold voter model; opinion dynamics; external influence; mean field approximation; diffusion approximation

1. Introduction

Human opinions and collective human behaviors have been studied for more than a century (Asch, 1956; Keynes, 2018; Le Bon, 2017; Mackay, 2015; Milgram & Fleissner, 1974; Schelling, 1971). Relatively recent are the mathematical analyses of these dynamics, which have been made possible due to seminal frameworks such as the voter model (VM) (Clifford & Sudbury, 1973; Holley & Liggett, 1975), DeGroot learning (DeGroot, 1974), and the naming game model (Baronchelli, Felici, Loreto, Caglioti, & Steels, 2006; Dall'Asta, Baronchelli, Barrat, & Loreto, 2006). A significant number of advances in opinion dynamics have their roots in statistical physics (Nyczka, Sznajd-Weron, & Cisło, 2012, Vazquez, & Eguíluz, 2008, p. 5). For details on the origins and evolution of this domain, the reader is referred to the comprehensive review by Castellano et al. (Castellano, Fortunato, & Loreto, 2009).

While various types of node-to-node interactions have received considerable attention (Cox & Durrett, 1991; Hegselmann & Krause et al., 2002; Liggett, 1994), the incorporation of external influences has received relatively little modeling scrutiny. One reason for this seems to be the associated loss of analytical tractability. There is evidence, however, that media can play an important role in opinion dynamics in many different contexts, for example, in climate change (Brulle, Carmichael, & Jenkins, 2012), and in electoral voting (Beck, Dalton, Greene, & Huckfeldt, 2002; Druckman, 2005). Furthermore, the commercial relevance of this phenomenon can be found in quantitative marketing, when multiple brands compete to sell their respective products via advertising. A theoretical examination of the effect of external influence is thus important. Existing work, however, is either not amenable to detailed mathematical analysis (Quattrociocchi, Caldarelli, & Scala, 2014), lacks generality (Fotouhi & Rabbat, 2013), or simply considers external influence as the initial condition for the dynamics, and not a part of the dynamics (Galam, 2008). We address these shortcomings by modeling external influence as a jump process and use the theory of jump-diffusion processes. Jump diffusion models are widely used in financial domains such as derivative pricing and risk management (Kou, 2007). We emphasize here that our jump process is

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not biased in one direction, as in other models of external influence (Przybyla, Sznajd-Weron, & Weron, 2014; Sznajd-Weron & Weron, 2003).

The organization of the paper is as follows: In section 2 we define our models, followed by our results and their interpretation in section 3. We finish with a high-level discussion in section 4.

2. Models

Let G be a $k-$regular undirected graph of N nodes, where each node is considered to be an individual agent, and the links represent social connections between the agents. Let $S(G)$ denote the set of nodes of G. Consider a binary opinion space, i.e., a node $x \in S(G)$ must have either opinion 0 or opinion 1 at time step n which is denoted by $s_x(n)$. Let A be the adjacency matrix, with the element A_{xy} being 1 if nodes x and y are connected, and 0 otherwise. The neighborhood of a node consists of the immediate nodes with which it is connected. *Consensus* is said to be reached when either all nodes have opinion 0 or opinion 1, and it is assumed that these states are absorbing.

2.1. *Jump voter model*

The jump voter model (JVM) is a discrete-time process that is updated according to two rules. At each time step, one of the following occurs:

Update Rule #1 (local updating). With probability $1-p$, a single node is randomly selected which then adopts the opinion of one of its neighbors chosen randomly. This local update rule affects (at most) one individual per time step.

Update Rule #2 (global updating). With probability p, we flip the opinions of multiple individuals of a given type. When such a probability p event is to occur, we generate a random variable Y with a truncated normal distribution on $[-1,1]$. The sign of Y indicates the opinion to be flipped (0 if $Y>0$, and 1 if $Y<0$), and the magnitude $|Y|$ indicates the fraction of the population to be flipped. The actual number of the given type to be flipped is given by the closest integer to YN. In the event that this number exceeds the current number of individuals with the chosen opinion, all of the available nodes are flipped and consensus results. Thus, letting Z denote the closest integer to YN, 0s will be flipped with probability 1/2 (if $Z<0$) and 1s will be flipped with probability 1/2 (if $Z>0$); the number to be flipped is $|Z|$.

The first update rule captures the node-to-node interactions of the "classical" VM. The second rule captures the more global external influence that makes several opinions flip simultaneously, a phenomenon that we call a *jump*. Call p the jump probability, and note that when $p=0$ the model reduces to a discrete-time version of the VM on a graph structure. The JVM is a discrete state space Markov chain, and the total number of opinion 1 nodes in the graph at time step n, denoted by $X^N(n)$, can be thought of as a global summary statistic of that Markov chain. More formally,

$$X^N(n) = \sum_{x\in S(G)} s_x(n).$$

The jump random variable Y is independent of the state of the process, and we restrict our attention to the case where the mean of Y is zero (the case of no bias in external influence). The variance of Y, denoted by v, can be thought of as the strength of the external influence, and we call it the jump variance. By varying v, we are able to see how the distribution of Z affects the results. Note that, at the time of a jump, the number ($|Z|$) of the chosen type to be flipped is independent of the current number of that type. This is analogous to the ecological notion of constant-yield harvesting – and different from the notion of constant-effort harvesting, which would have the number flipped proportional to the current frequency of that type. (A brief summary of all the model parameters is provided in Table 1.)

Table 1. Summary of the model parameters.

Parameter	Brief Description
N	Total population size.
k	Degree in a regular graph.
p	Jump probability at each time step.
Z (or Y)	Random variable whose absolute value gives the number (or fraction) of the population that jumps due to an external influence event, and whose sign determines which opinion gets flipped.
v	Jump variance, or strength of the external influence.
θ	Threshold parameter in the JTVM.
λ	Jump rate in the jump voter diffusion.

2.2. *Jump voter diffusion*

Here we consider a diffusion approximation for the (spatial) Jump Voter Model. The one-step transition probabilities in the absence of external influence ($p = 0$, i.e., Update Rule #1 only) are

$$\begin{aligned} P(X^N(n+1) = i+1 | X^N(n) = i) &= \sum_{\{x \in S(G): s_x = 0\}} \left(\frac{1}{N} \sum_{y \in S(G)} A_{xy} \frac{s_y}{k} \right) \\ P(X^N(n+1) = i-1 | X^N(n) = i) &= \sum_{\{x \in S(G): s_x = 1\}} \left(\frac{1}{N} \sum_{y \in S(G)} A_{xy} \frac{1-s_y}{k} \right) \end{aligned} \tag{1}$$

Using a mean-field approximation, in the spirit of that used by (Sood & Redner, 2005), the transition probabilities in equation (1) become,

$$\begin{aligned} P(X^N(n+1) = i+1 | X^N(n) = i) \approx \left(1 - \tfrac{i}{N}\right)\left(\tfrac{i}{N}\right) \\ P(X^N(n+1) = i-1 | X^N(n) = i) \approx \left(\tfrac{i}{N}\right)\left(1 - \tfrac{i}{N}\right). \end{aligned} \tag{2}$$

As $N \to \infty$, the scaled process $X^N([N^2 t])/N$ converges weakly to a diffusion process $X(t)$ with drift and diffusion terms

$$\begin{aligned} \mu(x) &= 0, \\ \sigma^2(x) &= 2x(1-x). \end{aligned} \tag{3}$$

(Above, $[N^2 t]$ is the integer part of $N^2 t$.) If we now introduce jumps ($p > 0$), and define $\lambda \equiv N^2 p$ (scaled jump rate), then for a small enough p and a large enough N, the JVM (Update Rules #1 and #2) can be approximated by a superposition of the diffusion derived above and a compound Poisson process, i.e., a jump-diffusion process given by,

$$dX(t) = \sqrt{2X(t)(1-X(t))}\, dW(t) + Y dN(t), \tag{4}$$

where $W(t)$ represents a Wiener process, and $N(t)$ represents a rate λ Poisson process. We denote by $g(x)$ the probability density function of the jump random variable Y, and also note that $g(x)$ will have support $[-1, 1]$. The generator of this process is the integrodifferential operator $\mathcal{L}$, defined by

$$\mathcal{L}f(x) = x(1-x)f''(x) + \lambda \int_{-\infty}^{+\infty} [f(x-y) - f(x)] g(y) dy. \tag{5}$$

Note here that as a result of using the mean field approximation, the jump voter diffusion given by equation (4) does not have a term containing the degree parameter k.

2.3. *Jump threshold voter model*

The jump threshold voter model (JTVM) is a discrete-time process that is updated according to two rules. At each time step, one of the following occurs:

Threshold Update Rule #1 With probability $1-p$, a single node is randomly selected. If the number of opposing opinions in the neighborhood of the selected node is greater than or equal to a threshold θ, then the opinion of the originally selected node is updated.

Threshold Update Rule #2 This rule is the same as Update Rule #2 for the Jump Voter Model above.

Note that the threshold θ applies only to the local updating. As with the JVM, the mean of Z is zero herein.

Table 1 summarises all the parameters discussed in this section.

3. Results

If $X(t)$ is a jump diffusion with generator $\mathcal{L}$, Itô's formula for jump processes (Protter, 2005) implies that

$$M(t) \equiv f(X(t)) - \int_0^t \mathcal{L}f(X(s))ds \tag{6}$$

is a martingale for any C^2 function $f(x)$. Application of the Optional Stopping Theorem (OST) to this martingale results in boundary value problems for both probability of consensus on 1 and expected value of the consensus time for the jump voter diffusion. Before that, we mathematically define the first hitting times of the two absorbing states (0 and 1) and the consensus time (τ)

$$\begin{aligned} T_0 &\equiv \inf\{t \geq 0 : X(t) \in (-\infty, 0]\}, \\ T_1 &\equiv \inf\{t \geq 0 : X(t) \in [1, \infty)\}, \\ \tau &\equiv \inf\{t \geq 0 : X(t) \in (-\infty, 0] \cup [1, \infty)\}. \end{aligned}$$

The probability of consensus on 1, $u(x) = P(T_1 < T_0 | X(0) = x)$, satisfies,

$$\begin{gathered} \mathcal{L}u(x) = 0, \\ u(x) = 0, \ \forall \ x \in (-\infty, 0], \\ u(x) = 1, \ \forall \ x \in [1, \infty). \end{gathered} \tag{7}$$

Similarly, the expected value of the consensus time (to all 0s or all 1s), $v(x) = E[\tau | X(0) = x]$, satisfies

$$\begin{gathered} \mathcal{L}v(x) = -1, \\ v(x) = 0, \ \forall \ x \in (-\infty, 0] \cup [1, \infty). \end{gathered} \tag{8}$$

3.1. *Jump voter model*

We begin by investigating the expected consensus time (to all 0s or all 1s) for the jump voter diffusion, and the probability that consensus is on opinion 1 (for a given initial fraction of 1). To determine the probability of consensus on 1, we use the generator of the jump voter diffusion from equation (5) in the boundary value problem (7), to obtain,

$$\begin{aligned} x(1-x)u''(x) + \lambda \int_{-\infty}^{+\infty} & u(x-y)g(y)dy - \lambda u(x) = 0 \\ u(x) &= 0, \quad \forall \quad x \in (-\infty, 0] \\ u(x) &= 1, \quad \forall \quad x \in [1, \infty). \end{aligned} \tag{9}$$

It is quite challenging to find an analytical solution for a variable-coefficient integrodifferential equation such as equation (9). Even a similar constant-coefficient equation requires imposing some structure on the function $g(x)$ to make a closed-form solution possible (Kou & Wang, 2004). We thus solve this problem numerically and compare the solution to Monte Carlo simulations of the JVM (Figure 1).

Similarly, if we expand equation (8) using the generator in equation (5), we obtain

$$\begin{aligned} x(1-x)v''(x) + \lambda\int_{-\infty}^{+\infty} v(x-y)g(y)dy - \lambda v(x) = -1 \\ v(x) = 0,\ \forall\ x \in (-\infty, 0] \cup [1, \infty). \end{aligned} \tag{10}$$

Solving the problem numerically, we find that the solution again closely matches the results of the Monte Carlo simulations of the JVM (Figure 2).

We make some remarks about interpreting the solutions of equations (9) and (10). The jump-diffusion is an asymptotic approximation to the JVM obtained by letting N to infinity. Therefore, the theoretical fixation probability and the theoretical expected consensus time for the JVM are approximated via the fixation probability and the expected consensus time for the jump-diffusion approximation (i.e., solutions of equations (9) and (10)), such that the quality of approximation improves as N increases. Note that the theoretical fixation probability for the JVM does not depend on N. On the other hand, the theoretical expected consensus time for the JVM is approximately equal to the expected consensus time for the jump-diffusion times N^2. Indeed, recall that the JVM is scaled as $X^N([N^2t])/N$ to obtain the jump-diffusion approximation, and therefore t units of time in the jump-diffusion approximation correspond to $[N^2t]$ units of time in the JVM. Moreover, note that the maximum value of the expected consensus time in the JVM is proportional to N^2.

We notice in Figure 2 that although our solution for the consensus time has properties that are qualitatively similar to those of the VM, the quantitative difference between the two is considerable. The two parameters, jump probability p and jump variance v, together determine the overall impact of the external influence, and we collectively refer to them as *jump parameters*. We find that even for fairly small jump parameter values, the expected consensus time differs dramatically between the VM and the JVM.

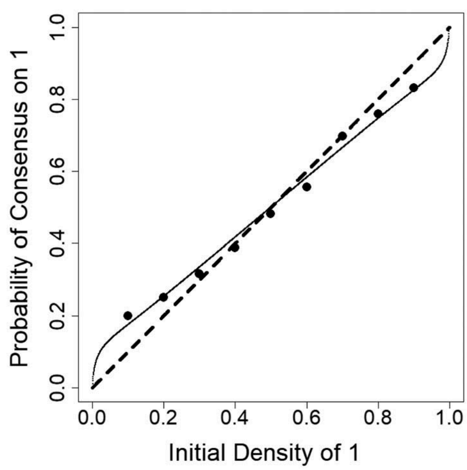


Figure 1. Probability of consensus on 1 for the JVM on a 25 × 25 lattice with $p = 1/(625 \times 10)$, with v = 0.03 (see Table 1 for the meaning of parameters). The probability is denoted by the solid curve, and corresponds to $u(x)$ which is obtained by numerically solving Equation (9); the dots show simulation results based on update rules in Section 2.1, where each point is obtained by averaging over 500 runs. For comparison, the probability of consensus on 1 for the VM on a 25 × 25 lattice is shown by the dashed line.

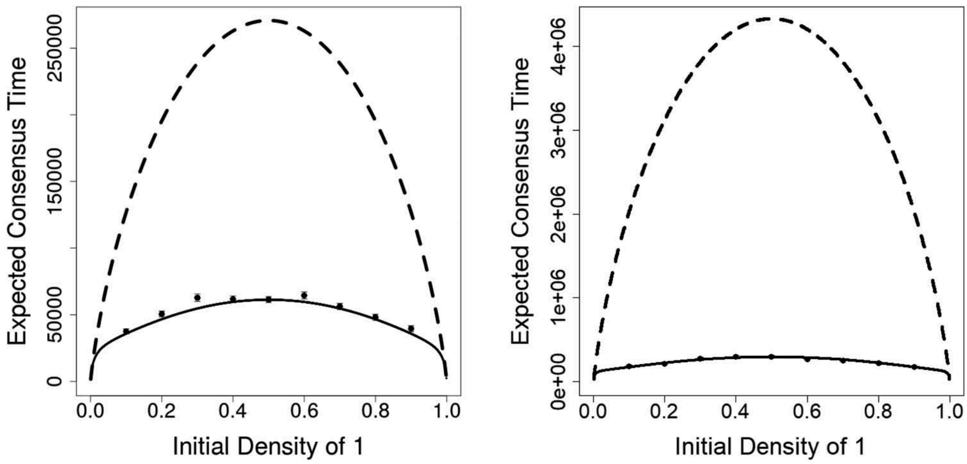


Figure 2. Expected consensus time for the JVM (left) on a 25×25 lattice with $p = 1/(625 \times 10)$ and (right) on a 50×50 lattice with $p = 1/(2500 \times 10)$, with $v = 0.03$ (see Table 1 for the meaning of parameters). The theoretical expected consensus time is denoted by the solid curve, and corresponds to plotting $v(x)N^2$ where $v(x)$ is obtained by numerically solving equation (10): $N = 625$ (left), $N = 2500$ (right); note that equation (10) produces the curve $v(x)$ for the jump-diffusion and the factor N^2 comes from the scaling of time in the discrete jump voter model: t units of time in the jump-diffusion approximation correspond to $[N^2t]$ units of time in the JVM. The dots show simulation results based on update rules in Section 2.1, where each point is obtained by averaging over 500 runs, and error bars indicate standard error of the mean. The maximum expected consensus time (i.e., consensus time at initial density 0.5) is 61267 (left) and 289003 (right). For comparison, the expected consensus time for the VM on a 25×25 lattice (left) and a 50×50 lattice (right) is shown by the dashed curve.

If we consider just the maximum value of consensus time (which occurs when the initial density of 1 is 0.5), we can plot it as a function of the two jump parameters (Figure 3). We see that the consensus time decreases rapidly as a function of p and v, especially at small values.

Consensus time in the VM is sensitive to the presence of jumps. Consensus time decreases in both jump parameters (p and v), and the rate of decrease of consensus time is very high at low jump

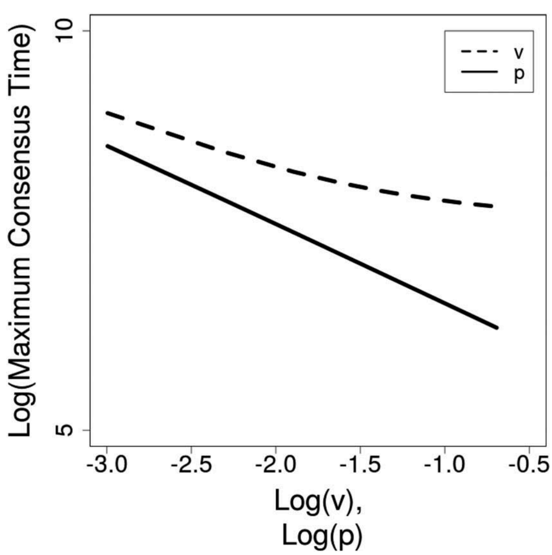


Figure 3. Maximum consensus time, on a log-log scale, for the jump voter diffusion as a function of the jump parameters individually, $N = 500$. (The maximum consensus time has been restricted to 10000 time steps for diagrammatic clarity.) The maximum consensus time decreases rapidly as p and v increase. For plotting the dependence on p, $v = 0.001$, and for plotting the dependence on v, $p = 0.001$ (see Table 1 for the meaning of parameters).

parameter values, and significantly drops as parameter values increase (Figure 3). Therefore, it is primarily the presence of jumps that appears to be a key factor for the consensus time. Note that the dependence of consensus time on jump parameter p exhibits a power-law structure (Figure 3). Overall, jumps expedite consensus, introducing little skew in addition to that inherently present due to the initial densities. Therefore, jumps may be an important ingredient to consider in opinion dynamics models based on the VM.

3.2. Jump threshold voter model

In this section, we study the probability of consensus on 1 and consensus time for the JTVM. The results are obtained through Monte Carlo simulations and are shown in Figure 4.

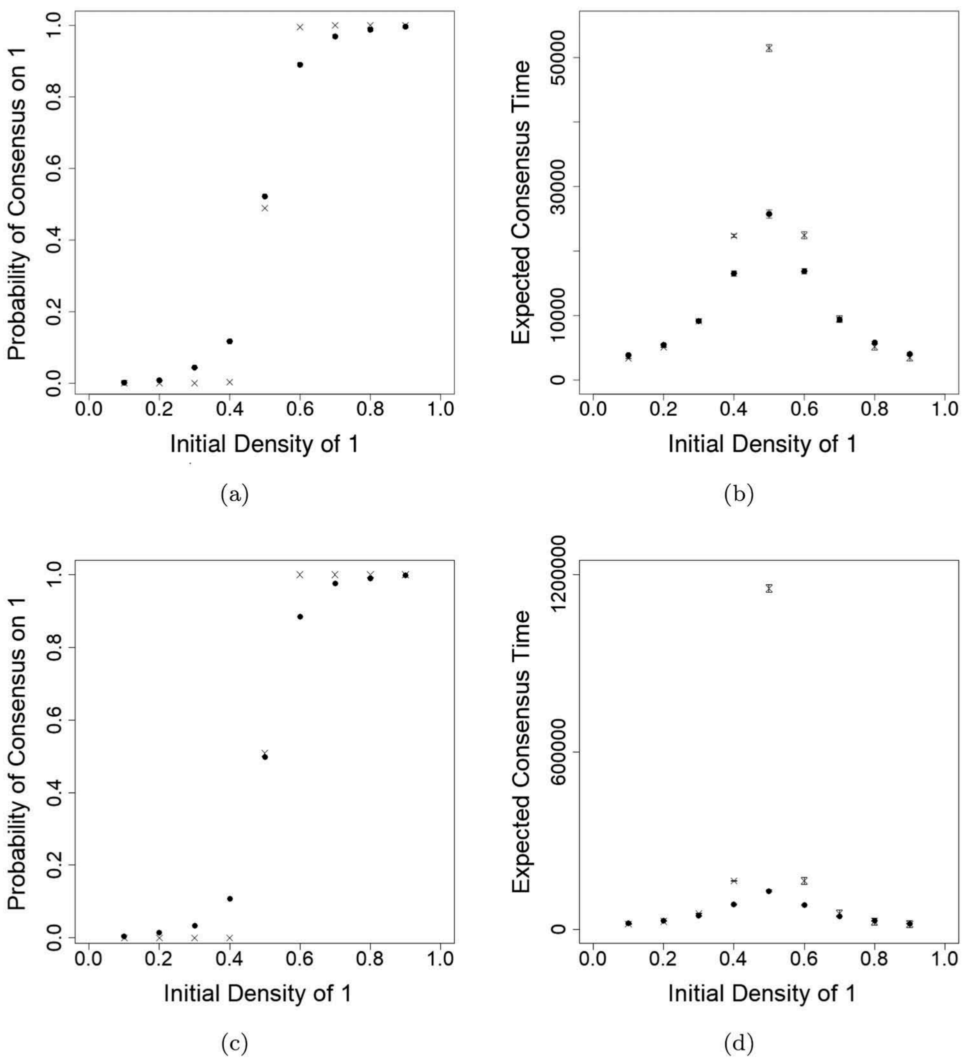


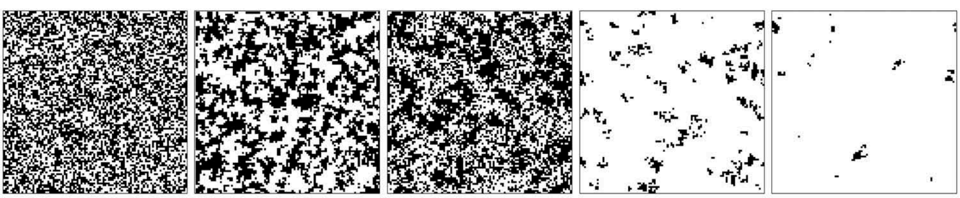
Figure 4. Probability of consensus on 1 ((a) and (c)), and expected consensus time ((b) and (d)) for the JTVM (●) in comparison with the TVM (×), (top row) on a 25 × 25 with $p = 1/(625 \times 10)$ and (bottom row) on a 50 × 50 lattice with $p = 1/(2500 \times 10)$, with $v = 0.03, \theta = 2$ (see Table 1 for meaning of parameters). Each point is based on 1000 runs, and error bars indicate standard error of the mean.

The spatial structure of the dynamics yields further insight into the model behavior. Typical spatial results are shown in Figure 5. It is known that the evolution of clusters in the TVM is characterized by motion by mean curvature (Castellano et al., 2009; Dall'Asta & Castellano, 2007). The introduction of jumps to this model plays the role of disrupting the clustering sporadically. This can be seen in Figure 5b.

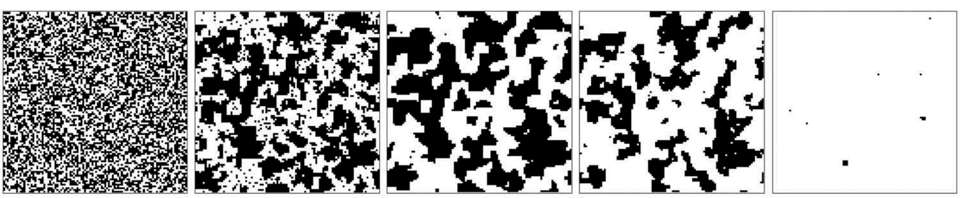
The JTVM is similar, for the probability of consensus on 1 *and the consensus time, to the TVM at low initial minority densities*. The differences between the TVM and the JTVM become prominent only at initial minority densities greater than 0.4 approximately, which we call the *critical density*. The probability of consensus on 1 for the TVM exhibits behavior similar to a step function (Figure 4), where consensus is almost always reached on the opinion in the majority at the start of the process. The TVM thus amplifies the advantage held by a particular opinion type due to a higher initial density. Comparing the probability of consensus on 1 in this case to that for the JTVM, we notice that the probabilities become less extreme at initial minority densities greater than the critical density. Therefore, at initial minority densities greater than the critical density, jumps help moderate the amplification advantage inherent to the TVM. Next, we assess the effect of jumps on the consensus time. Jumps have an effect of increasing fluctuations in the opinion density. This effect appears more pronounced at initial minority densities higher than the critical density, as we obtain a quicker consensus in that regime.

Overall, for initial minority densities lower than the critical density, the clustering effect is too strong to be disrupted by jumps. However, once the initial minority density exceeds a certain value, the disrupting effects of jumps begin to counter the clustering effect inherent to the TVM.

Jumps expedite consensus in the TVM less than the VM. We notice, based on comparing Figures 2 and 4, that jumps reduce the consensus time significantly more in the VM than in the TVM. This suggests that the TVM is more robust to an external influence. This is again attributable to motion by mean curvature in the TVM. Consider the extreme case, for both the JVM and JTVM, where the jump causes a $0 \rightarrow 1$ flip at a node in the interior of a 0 cluster. In the JVM, there is a non-zero probability that the 1 in the interior can, in turn, cause a $0 \rightarrow 1$ flip at one of its neighboring nodes. But in the TVM, the probability that this opinion 1 node can flip any of its opinion 0 neighbors is zero. Therefore, the cluster patterns of the VM tend to be more vulnerable to jumps as compared to those of the TVM. This resistance serves as a possible explanation for the higher robustness of the



(a) JVM



(b) JTVM

Figure 5. Snapshots of the evolution of (a) the JVM, and (b) the JTVM on a 100×100 lattice, with $p = 1/(10000 \times 5)$, $v = 0.03$, $\theta = 2$ (see Table 1 for the meaning of parameters). Initial density of 1 is 0.5 for both models. The vertical and horizontal straight edges in (b) are features of the threshold voter model with interactions between the 4 nearest neighbors and threshold 2; in fact, vertical and horizontal bands become frozen. The jump component in the JTVM, however, breaks down these straight edges over time.

TVM to jumps with regards to the consensus time. This phenomenon can also be observed in Figure 5: Notice the similarity in the cluster pattern in the second and fourth panels of Figure 5b despite the occurrence of jumps between them. On the contrary, no such trend is seen in Figure 5a.

4. Discussion

In this work, we have developed extensions of the voter model (VM) and the threshold voter model (TVM), that incorporate an external influence in the form of many opinions shifting in the same direction simultaneously. Most of the existing literature on opinion dynamics focuses solely on opinion evolution under influences internal to the system. This work provides a systematic study of opinion dynamics under an additional external influence. We approximated the jump voter model (JVM) by means of a jump diffusion process. This approach allowed us to analytically determine the probability of reaching consensus on opinion 1 and the mean consensus time. For the jump threshold voter model (JTVM), we relied on simulations to determine these quantities.

Note that jumps expedite consensus more in the VM than in the TVM. Thus, in a society where agents update their opinion if the pressure from their neighborhood is sufficient (based on a threshold parameter), external influence has a lesser effect on consensus time than in a society where agents update their opinions by randomly sampling from their neighborhood.

The model considered here is similar in some respects to the so-called noisy voter model (Carro, Toral, & San Miguel, 2016; Granovsky & Madras, 1995) where, instead of having group opinion changes, only one individual is allowed to flip its state in an update. In the noisy voter model, spontaneous flips prevent the system from reaching the absorbing consensus states (in the limit of large N). For finite N, the effect of noise is that the time to reach consensus is increased (more precisely, it grows exponentially with N), and the fixation probability tends to a constant ($1/2$), independent of the initial density of 1. These results are in sharp contrast with those found in this paper, where instead noise favors consensus. In the noisy voter model, the individual whose opinion will be randomly flipped is chosen randomly, and hence with frequency dependence – i.e., the minority type is less likely to be flipped from noise. This pushes the system toward equal frequencies and explains the lengthened consensus times and lack of dependence on the initial configuration when N is large. On the other hand, given that a jump will occur, the JVM has probability $1/2$ of flipping a fixed (randomly chosen) number Z of the minority type, no matter how rare it is (possibly removing all minority members in one flip). When the majority type is flipped, it results in the same numerical loss for the same value Z, but this is a much smaller loss in terms of frequency.

This work opens up multiple interesting directions that may be explored further. The domain of network science is currently expanding very rapidly, and one natural extension of our work is to study our models on a heterogeneous graph structure such as a scale-free network. Such work could lead to pragmatic insights since scale-free networks have been shown to be ubiquitous in various real-world social systems (Barabási and Albert, 1999). The effect of "stubborn" individuals (i.e., individuals with fixed opinions) is another concern, and the interacting effects of stubbornness and media influence could be explored by combining our approach with existing work on opinion dynamics with stubbornness (Perez-Llanos, Pinasco, Saintier, & Silva, 2018).

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