Advanced Skills

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\[ y = \sum_{n=1}^{\infty} 2Z + 4PQ^2 \]

\[ C = A \cap B \]

\[ a = b \pm 4 \]

\[ y = \sqrt{x^3 + 2x} - \left( \frac{5x}{x^2} \right) \]

\[ y = mx + c \]
Introduction

Fourier Transform theory is essential to proper understanding of the mathematics behind image processing and remote sensing.

A Math bit:

*The Fourier Transform can either be considered as an expansion in terms of an orthogonal bases set (sine and cosine), or a shift of space from “real space” to ‘reciprocal space’.*

Actually these two concepts are mathematically identical although they are often used in very different physical situations.

The aim of this booklet is to cover the background Fourier Theory required for the graduate level image processing and remote sensing courses. This booklet was produced by King’s College London (Department of Physics) and revised into current form by the University of Edinburgh (Department of Physics).

Further details of Fourier Transforms can be found in: “Introduction to the Fourier Transform and its Applications” by Bracewell.

1.1 Notation

Unlike many mathematical fields of science, Fourier Transform theory does not have a well-defined set of standard notations. The notation maintained throughout will be:

\[ x, y \rightarrow \text{Real Space co-ordinates} \]
\[ u, v \rightarrow \text{Frequency Space co-ordinates} \]

and lower case functions (eg \( f(x) \)), being a real space function and upper case functions (\( F(u) \)), being the corresponding Fourier transform, thus:

\[ F(u) = F\{f(x)\} \]
\[ F(x) = F^{-1}\{F(u)\} \]

where \( F \) is the Fourier Transform operator.

The character \( i \) will be used to denote \( \sqrt{-1} \) it should be noted that this character differs from the conventional \( i \) (or \( j \)). This slightly odd convention and is to avoid confusion when the digital version of the Fourier Transform is discussed in some courses since then \( i \) and \( j \) will be used as summation variables.

Two special functions will also be employed, these being sinc() defined as,

\[ \text{sinc}(x) = \frac{\sin(x)}{x} \]
giving \( \text{sinc}(0) = 1 \) and \( \text{sinc}(x_0) = 0 \) at \( x_0 = \pm \pi, \pm 2\pi, \ldots \), as shown in figure 1:

Secondly, the top hat function \( \Pi(x) \) is given by,

\[
\Pi(x) = \begin{cases} 
1 & \text{for } |x| \leq \frac{1}{2} \\
0 & \text{else}
\end{cases}
\]

being a function of unit height and width centered about \( x = 0 \), and is shown in figure 2.

Figure 1: The \( \text{sinc}(x) \) function.

Figure 2: The \( \Pi(x) \) function.
2 The Fourier Transform

The definition of a one dimensional continuous function, denoted by \( f(x) \), the Fourier transform is defined by:

\[
F(u) = \int_{-\infty}^{\infty} f(x) \exp(-2\pi iux) \, dx
\]

(1)

with the inverse Fourier transform defined by;

\[
f(x) = \int_{-\infty}^{\infty} F(u) \exp(i2\pi ux) \, du
\]

(2)

where it should be noted that the factors of \( 2\pi \) are incorporated into the transform kernel\(^1\).

Some insight to the Fourier transform can be gained by considering the case of the Fourier transform of a real signal \( f(x) \). In this case the Fourier transform can be separated to give,

\[
F(u) = F_r(u) + iF_i(u)
\]

(3)

where we have,

\[
F_r(u) = \int_{-\infty}^{\infty} f(x) \cos(2\pi ux) \, dx
\]

\[
F_i(u) = -\int_{-\infty}^{\infty} f(x) \sin(2\pi ux) \, dx
\]

So the real part of the Fourier transform is the decomposition of \( f(x) \) in terms of cosine functions, and the imaginary part a decomposition in terms of sine functions. The \( u \) variable in the Fourier transform is interpreted as a frequency, for example if \( f(x) \) is a sound signal with \( x \) measured in seconds then \( F(u) \) is its frequency spectrum with \( u \) measured in Hertz (\( s^{-1} \)).

**NOTE:** Clearly \( (ux) \) must be dimensionless, so if \( x \) has dimensions of time then \( u \) must have dimensions of \( time^{-1} \).

This is one of the most common applications for Fourier Transforms where \( f(x) \) is a detected signal (for example a sound made by a musical instrument), and the Fourier Transform is used to give the spectral response.

2.1 Properties of the Fourier Transform

The Fourier transform has a range of useful properties, some of which are listed below. In most cases the proof of these properties is simple and can be formulated by use of equation (3) and (4). The proofs of many of these properties are given in the questions and solutions at the back of this booklet.

**Linearity:** The Fourier transform is a linear operation so that the Fourier transform of the sum of two functions is given by the sum of the individual Fourier transforms. Therefore,

\[
F \{ af(x) + bg(x) \} = aF(u) + bG(u)
\]

(4)

\(^1\)There are various definitions of the Fourier transform that puts the \( 2\pi \) either inside the kernel or as external scaling factors. The difference between them whether the variable in Fourier space is a “frequency” or “angular frequency”. The difference between the definitions are clearly just a scaling factor. The optics and digital Fourier applications the \( 2\pi \) is usually defined to be inside the kernel but in solid state physics and differential equation solution the \( 2\pi \) constant is usually an external scaling factor.
where $F(u)$ and $G(u)$ are the Fourier transforms of $f(x)$ and and $g(x)$ and $a$ and $b$ are constants. This property is central to the use of Fourier transforms when describing linear systems.

**Complex Conjugate:** The Fourier transform of the Complex Conjugate of a function is given by

$$F \{f^*(-x)\} = F^*(-u)$$

where $F(u)$ is the Fourier transform of $f(x)$.

**Forward and Inverse:** We have that

$$F \{F(u)\} = f(-x)$$

so that if we apply the Fourier transform twice to a function, we get a spatially reversed version of the function. Similarly with the inverse Fourier transform we have that,

$$F^{-1} \{f(x)\} = F(-u)$$

so that the Fourier and inverse Fourier transforms differ only by a sign.

**Differentials:** The Fourier transform of the derivative of a functions is given by

$$F \left\{ \frac{df(x)}{dx} \right\} = i2\pi u F(u)$$

and the second derivative is given by

$$F \left\{ \frac{d^2f(x)}{dx^2} \right\} = -(2\pi u)^2 F(u)$$

**Power Spectrum:** The Power Spectrum of a signal is defined by the modulus square of the Fourier transform, being $|F(u)|^2$. This can be interpreted as the power of the frequency components. Any function and its Fourier transform obey the condition that

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |F(u)|^2 \, du$$

which is frequently known as Parseval’s Theorem. If $f(x)$ is interpreted at a voltage, then this theorem states that the power is the same whether measured in real (time), or Fourier (frequency) space.
Symmetry: If \( f(x) \) is real, then equation (1) can be written in terms of its real and imaginary parts as:

\[
F(u) = F_r(u) + \imath F_i(u)
\]

where,

\[
F_r(u) = \int_{-\infty}^{\infty} f(x) \cos(2\pi ux) dx
\]

\[
F_i(u) = \int_{-\infty}^{\infty} f(x) \sin(2\pi ux) dx
\]

As \( \cos(x) \) is a symmetric function then so is \( F_r(x) \). As \( \sin(x) \) is antisymmetric, so also is \( F_i(x) \).

Translation: Translation in real-space is equivalent to multiplication by a phase-factor in Fourier space.

\[
F\{f(x+a)\} = \exp(\imath 2\pi ua)F\{f(x)\}
\]

Here’s the proof if you don’t believe it:

\[
F\{f(x+a)\} = \int_{-\infty}^{\infty} f(x+a) \exp(-\imath 2\pi ux) dx
\]

\[
= \int_{-\infty}^{\infty} f(x') \exp(-\imath 2\pi u(x'-1)) dx
\]

\[
= \exp(\imath 2\pi ua)F\{f(x)\}
\]

Magnification: Magnification in real-space is equivalent to reduction and amplitude reduction in Fourier-Space.

\[
F\{f(ax)\} = \frac{1}{a} F\left(\frac{u}{a}\right)
\]

Periodicity: The Fourier transform of a periodic function with period \( L \) is a regular array of impulses having separation \( 1/L \).

2.2 Two Dimensional Fourier Transform

As Image processing and remote sensing use two-dimensional images, we introduce the two-dimensional Fourier transform of a continuous function \( f(x,y) \) as:
\[ F(u, v) = \iint f(x, y) \exp(-i 2\pi(ux + vy)) \, dx \, dy \]

with the inverse Fourier transform defined by;

\[ f(x, y) = \iint F(u, v) \exp(i 2\pi(ux + vy)) \, du \, dv \] (12)

where the limits of integration are taken from \(-\infty \rightarrow \infty^3\).

Again for a real two dimensional function \( f(x, y) \), the Fourier transform can be considered as the decomposition of a function into its sinusoidal components. If \( f(x, y) \) is considered to be an image with the “brightness” of the image at point \( (x_0, y_0) \) given by \( f(x_0, y_0) \), then variables \( x, y \) have the dimensions of length. In Fourier space the variables \( u, v \) have therefore the dimensions of inverse length, which is interpreted as Spatial Frequency.

NOTE: Typically \( x \) and \( y \) are measured in mm so that \( u \) and \( v \) have are in units of mm\(^{-1}\) also referred to at lines per mm.

The Fourier transform can then be taken as being the decomposition of the image into two dimensional sinusoidal spatial frequency components. This property will be examined in greater detail the relevant courses.

The properties of one the dimensional Fourier transforms covered in the previous section convert into two dimensions. Clearly the derivatives then become

\[ F \left\{ \frac{\partial f(x, y)}{\partial x} \right\} = i 2\pi u F(u, v) \] (13)

and with

\[ F \left\{ \frac{\partial f(x, y)}{\partial y} \right\} = i 2\pi v F(u, v) \] (14)

yielding the important result that,

\[ F \left\{ \nabla^2 f(x, y) \right\} = -(2\pi w)^2 F(u, v) \] (15)

where we have that \( w^2 = u^2 + v^2 \). So that taking the Laplacian of a function in real space is equivalent to multiplying its Fourier transform by a circularly symmetric quadratic of \(-4\pi^2 w^2\).

The two dimensional Fourier Transform \( F(u, v) \), of a function \( f(x, y) \) is a separable operation, and can be written as,

\[ F(u, v) = \int P(u, y) \exp(-i 2\pi vy) \, dv \] (16)

where

\[ P(u, y) = \int f(x, y) \exp(-i 2\pi ux) \, dx \] (17)

where \( P(u, y) \) is the Fourier Transform of \( f(x, y) \) with respect to \( x \) only. This property of separability will be considered in greater depth with regards to digital images and will lead to an implementation of two dimensional discrete Fourier Transforms in terms of one dimensional Fourier Transforms.
3 Dirac Delta Function

A frequently used concept in Fourier theory is that of the Dirac Delta Function, which is somewhat abstractly defined as:

\[
\delta(x) = 0 \quad \text{for } x \neq 0
\]

\[
\int_{-\infty}^{\infty} \delta(x) \, dx = 1
\]  \hspace{1cm} (1)

This can be thought of as a very "tall-and-thin" spike with unit area located at the origin, as shown in figure 3.

\[\delta(x)\]

\[\text{Figure 1: The \(\delta\)-function.}\]

**NOTE:** The \(\delta\)-functions should **not** be considered to be an "infinitely high" spike of "zero" width since it scales as:

\[
\int_{-\infty}^{\infty} a \delta(x) \, dx = a
\]

where \(a\) is a constant.

The **Delta Function** is not a true function in the analysis sense and if often called an improper function. There are a range of definitions of the Delta Function in terms of proper function, some of which are:

\[
\Delta_{\varepsilon}(x) = \frac{1}{\varepsilon \sqrt{\pi}} \exp\left(\frac{-x^2}{\varepsilon^2}\right)
\]

\[
\Delta_{\varepsilon}(x) = \frac{1}{\varepsilon} \Pi \left(1 + \frac{x}{\varepsilon}\right)
\]

\[
\Delta_{\varepsilon}(x) = \frac{1}{\varepsilon} \text{sinc} \left(\frac{x}{\varepsilon}\right)
\]

being the Gaussian, Top-Hat and Sinc approximations respectively. All of these expressions have the property that,

\[
\int_{-\infty}^{\infty} \Delta_{\varepsilon}(x) \, dx = 1 \quad \forall \varepsilon
\]  \hspace{1cm} (2)

and we may form the approximation that,

\[
\delta(x) = \lim_{\varepsilon \to 0} \Delta_{\varepsilon}(x)
\]  \hspace{1cm} (3)

which can be interpreted as making any of the above approximations \(\Delta_{\varepsilon}(x)\) a very "tall-and-thin" spike with unit area.
In the field of optics and imaging, we are dealing with two dimensional distributions, so it is especially useful to define the Two Dimensional Dirac Delta Function, as,

\[
\delta(x, y) = 0 \quad \text{for } x \neq 0 \land y \neq 0
\]

\[
\iint \delta(x, y) \, dx \, dy = 1
\]

(4)

which is the two dimensional version of the \(\delta(x)\) function defined above, and in particular:

\[
\delta(x, y) = \delta(x) \delta(y)
\]

(5)

This is the two dimensional analogue of the impulse function used in signal processing. In terms of an imaging system, this function can be considered as a single bright spot in the centre of the field of view, for example a single bright star viewed by a telescope.

3.1 Properties of the Dirac Delta Function

Since the Dirac Delta Function is used extensively, and has some useful, and slightly peculiar properties, it is worth considering these are this point. For a function \(f(x)\), being integrable, then we have that

\[
\int_{-\infty}^{\infty} \delta(x) f(x) \, dx = f(0)
\]

(6)

which is often taken as an alternative definition of the Delta function. This says that integral of any function multiplied by a \(\delta\)-function located about zero is just the value of the function at zero. This concept can be extended to give the Shifting Property, again for a function \(f(x)\), giving,

\[
\int_{-\infty}^{\infty} \delta(x - a) f(x) \, dx = f(a)
\]

(7)

where \(\delta(x - a)\) is just a \(\delta\)-function located at \(x = a\) as shown in figure 4.

![Figure 2: Shifting property of the \(\delta\)-function.](image)

In two dimensions, for a function \(f(x, y)\), we have that,

\[
\iint \delta(x - a, y - b) f(x, y) \, dx \, dy = f(a, b)
\]

(8)

where \(\delta(x - a, y - b)\) is a \(\delta\)-function located at position \(a, b\). This property is central to the idea of convolution, which is used extensively in image formation theory, and in digital image processing.
The Fourier transform of a Delta function is can be formed by direct integration of the definition of the Fourier transform, equation (3) and the property in (25) above, then we get that,

$$F \{ \delta(x) \} = \int_{-\infty}^{\infty} \delta(x) \exp(-i2\pi ux) \, dx = \exp(0) = 1$$  \hspace{1cm} (9)

and then by the Shifting Theorem, equation (26), we get that,

$$F \{ \delta(x-a) \} = \exp(i2\pi au)$$  \hspace{1cm} (10)

so that the Fourier transform of a shifted Delta Function is given by a phase ramp. It should be noted that the modulus squared of (29) is

$$|F \{ \delta(x-a) \}|^2 = |\exp(-i2\pi au)|^2 = 1$$

saying that the power spectrum a Delta Function is a constant independent of its location in real space.

Now noting that the Fourier transform is a linear operation, then if we consider two Delta Function located at $\pm a$, then from equation (6) the Fourier transform gives,

$$F \{ \delta(x-a) + \delta(x+a) \} = \exp(i2\pi au) + \exp(-i2\pi au) = 2\cos(2\pi au)$$  \hspace{1cm} (11)

while if we have the Delta Function at $x = -a$ as negative, then we also have that,

$$F \{ \delta(x-a) - \delta(x+a) \} = \exp(i2\pi au) - \exp(-i2\pi au) = 2i\sin(2\pi au)$$  \hspace{1cm} (12)

Noting the relations between forward and inverse Fourier transform we then get the two useful results that

$$F \{ \cos(2\pi ax) \} = \frac{1}{2} [\delta(u + a) + \delta(u - a)]$$  \hspace{1cm} (13)

and that

$$F \{ \sin(2\pi ax) \} = \frac{1}{2i} [\delta(u + a) - \delta(u - a)]$$  \hspace{1cm} (14)

So that the Fourier transform of a cosine or sine function consists of a single frequency given by the period of the cosine or sine function as would be expected.

### 3.2 The Infinite Comb

If we have an infinite series of Delta functions at a regular spacing of $\Delta x$, this is described as an Infinite Comb. The the expression for a Comb is given by,

$$\text{Comb}_{\Delta x}(x) = \sum_{i=\infty}^{\infty} \delta(x - i\Delta x)$$  \hspace{1cm} (15)

A short section of such a Comb is shown in figure 5.

Since the Fourier transform is a linear operation then the Fourier transform of the infinite comb is the sum of the Fourier transforms of shifted Delta functions, which from equation (29) gives,

$$F \{ \text{Comb}_{\Delta x}(x) \} = \sum_{i=-\infty}^{\infty} \exp(-i2\pi i\Delta xu)$$  \hspace{1cm} (16)
Figure 3: Infinite Comb with separation $\Delta x$

Now the exponential term,

$$\exp(-i2\pi\Delta xu) = 1 \quad \text{when } 2\pi\Delta xu = 2\pi u$$

so that:

$$\sum_{k=-\infty}^{\infty} \exp(-i2\pi\Delta xu) \rightarrow \infty \quad \text{when } u = \frac{2\pi}{\Delta x}$$

$$= 0 \quad \text{else}$$

which is an infinite series of $\delta$-function at a separation of $\Delta u = \frac{1}{\Delta x}$. So that an Infinite Comb Fourier transforms to another Infinite Comb or reciprocal spacing,

$$F\{\text{Comb}_{\Delta x}(x)\} = \text{Comb}_{\Delta u}(u) \quad \text{with } \Delta u = \frac{1}{\Delta x} \quad (17)$$

This is an important result used in Sampling Theory in the DIGITAL IMAGE ANALYSIS and IMAGE PROCESSING I courses.

Figure 4: Fourier Transform of comb function.
4 Symmetry Conditions

When we take the the Fourier Transform of a real function, for example a one-dimensional sound signal or a two-dimensional image we obtain a complex Fourier Transform. This Fourier Transform has special symmetry properties that are essential when calculating and/or manipulating Fourier Transforms.

4.1 One-Dimensional Symmetry

Firstly consider the case of a one dimensional real function \( f(x) \), with a Fourier transform of \( F(u) \). Since \( f(x) \) is real then from previous we can write

\[
F(u) = F_r(u) + iF_i(u)
\]

where the real and imaginary parts are given by the cosine and sine transforms to be

\[
F_r(u) = \int f(x) \cos(2\pi ux) \, dx
\]
\[
F_i(u) = -\int f(x) \sin(2\pi ux) \, dx
\]

now \( \cos() \) is a symmetric function and \( \sin() \) is an anti-symmetric function, as shown in figure 7, so that:

\[
F_r(u) \quad \text{is Symmetric}
\]
\[
F_i(u) \quad \text{is Anti-symmetric}
\]

which can be written out explicitly as,

\[
F_r(u) = F_r(-u)
\]
\[
F_i(u) = -F_i(-u)
\]

![Cosine and Sine Functions](image)

Figure 1: Symmetry properties of \( \cos() \) and \( \sin() \) functions

The power spectrum is given by

\[
|F(u)|^2 = F_r(u)^2 + F_i(u)^2
\]
so that if the real and imaginary parts obey the symmetry property given in equation (38), then clearly the power spectrum is also symmetric with

$$|F(u)|^2 = |F(-u)|^2$$

(3)

so when the power spectrum of a signal is calculated it is normal to display the signal from $0 \rightarrow u_{\text{max}}$ and ignore the negative components.

### 4.2 Two-Dimensional Symmetry

In two dimensional we have a real image $f(x,y)$, and then as above the Fourier transform of this image can be written as,

$$F(u,v) = F_r(u,v) + iF_i(u,v)$$

(4)

where after expansion of the exp() functions into cos() and sin() functions we get that

$$F_r(u,v) = \iint f(x,y) \left[ \cos(2\pi ux) \cos(2\pi vy) - \sin(2\pi ux) \sin(2\pi vy) \right] \, dx \, dy$$

and that,

$$F_i(u,v) = \iint f(x,y) \left[ \cos(2\pi ux) \sin(2\pi vy) + \sin(2\pi ux) \cos(2\pi vy) \right] \, dx \, dy$$

In this case the symmetry properties are more complicated, however we say that the real part is symmetric and the imaginary part is anti-symmetric, where in two dimensions the symmetry conditions are given by,

$$F_r(u,v) = F_r(-u,-v)$$

(5)

$$F_r(-u,-v) = F_r(u,v)$$

for the real part of the Fourier transform, and

$$F_i(u,v) = -F_i(-u,v)$$

(6)

$$F_i(-u,v) = -F_i(u,v)$$

for the imaginary part. Similarly the two dimensional power spectrum is also symmetric, with

$$|F(u,v)|^2 = |F(-u,-v)|^2$$

$$|F(-u,v)|^2 = |F(u,-v)|^2$$

(7)

This symmetry condition is shown schematically in figure 8 which shows a series of symmetric points.

These symmetry properties have a major significance in the digital calculation of Fourier transforms and the design of digital filters, which is discussed in greater detail in the relevant courses.
Figure 2: Symmetry in two dimensions
5 Convolution of Two Functions

The concept of convolution is central to Fourier theory and the analysis of Linear Systems. In fact the convolution property is what really makes Fourier methods useful. In one dimension the convolution between two functions, \( f(x) \) and \( h(x) \) is defined as:

\[
g(x) = f(x) \ast h(x) = \int_{-\infty}^{\infty} f(s)h(x-s)\,ds
\]  

(1)

where \( s \) is a dummy variable of integration. This operation may be considered the area of overlap between the function \( f(x) \) and the spatially reversed version of the function \( h(x) \). The result of the convolution of two simple one dimensional functions is shown in figure 9.

![Convolution of two simple functions](image)

Figure 1: Convolution of two simple functions.

The Convolution Theorem relates the convolution between the real space domain to a multiplication in the Fourier domain, and can be written as:

\[
G(u) = F(u)H(u)
\]

(2)

where

\[
G(u) = F\{g(x)\} \\
F(u) = F\{f(x)\} \\
H(u) = F\{h(x)\}
\]

This is the most important result in this booklet and will be used extensively in all three courses. This concept may appear a bit abstract at the moment but there will be extensive illustrations of convolution throughout the courses.
5.1 Simple Properties

The convolution is a linear operation which is distributive, so that for three functions \( f(x) \), \( g(x) \) and \( h(x) \) we have that

\[
(f(x) \ast (g(x) \ast h(x))) = ((f(x) \ast g(x)) \ast h(x))
\]  

(3)

and commutative, so that

\[
f(x) \ast h(x) = h(x) \ast f(x)
\]  

(4)

If the two functions \( f(x) \) and \( h(x) \) are of finite extent, (are zero outwith a finite range of \( x \)), then the extent (or “width”) of the convolution \( g(x) \) is given by the sum of the widths the two functions. For example if figure 9 both \( f(x) \) and \( h(x) \) non-zero over the finite range \( x = \pm 1 \) which the convolution \( g(x) \) is non-zero over the range \( x = \pm 2 \). This property will be used in optical image formation and in the practical implication of convolution filters in digital image processing.

The special case of the convolution of a function with a \( \text{Comb}(x) \) function results in replication of the function at the comb spacing as shown in figure 10. Clearly if the extent of the function is less than the comb spacing, as shown in this figure, the replications are separated, while if the extent of the function is greater than the comb period, overlap of adjacent replications will occur. This operation is central to sampling theory, and image formation and will be discussed in details in the relevant courses. This idea is also central to Solid State Physics where the electron density of a unit cell is convolved with the lattice sites.

![Figure 2: Convolution of function with comb of \( \delta \)-functions.](image)

5.2 Two Dimensional Convolution

As with Fourier Transform the extension to two-dimensions is simple with,

\[
g(x,y) = f(x,y) \ast h(x,y) = \int f(s_x,t) h(x-s_x,y-t) \, ds \, dt
\]  

(5)

which in the Fourier domain gives the important result that,

\[
G(u,v) = F(u,v) H(u,v)
\]  

(6)

This relation is fundamental to both optics and image processing and will be used extensively in the both courses.

The most important implication of the Convolution Theorem is that,

\[
\text{Multiplication in Real Space} \iff \text{Convolution in Fourier Space}
\]

\[
\text{Convolution in Real Space} \iff \text{Multiplication in Fourier Space}
\]

which is a Key Result.
6 Correlation of Two Functions

A closely related operation to Convolution is the operation of Correlation of two functions. In Correlation two function are shifted and the area of overlap formed by integration, but this time without the spatial reversal involved in convolution. The Correlation between two function $f(x)$ and $h(x)$ is given by

$$c(x) = f(x) \ast h(x) = \int_{-\infty}^{\infty} f(s) \overline{h(s-x)} \, ds  \tag{7}$$

where $\overline{h(x)}$ is the complex conjugate of $h(x)$\(^1\). This operation is shown for two simple functions in figure 11. Comparison between the convolution in figure 9 and the correlation shown that the only difference is that the second function is not spatially reversed and the direction of the shift is changed.

![Figure 3: Correlation of two simple functions.](image)

Of more importance, if we consider $f(x)$ to be the “signal” and $h(x)$ to be the “target” then we see that the correlation gives a peak where the “signal” matches the “target”. This gives the basis of the simples method of target detection\(^2\).

In the Fourier Domain the Correlation Theorem becomes

$$C(u) = F(u)H^*(u) \tag{8}$$

where

$$C(u) = F \{c(x)\}, \quad F(u) = F \{f(x)\}, \quad H(u) = F \{h(x)\}$$

\(^1\)It should be noted that for a real function complex conjugation does not effect the function, so if both $f(x)$ and $h(x)$ are real then the Convolution and Correlation differ only by a change of sign, which represents the spatial reversal on one of the functions.

\(^2\)The two-dimensional version of this is considered in question 9.
It should be noted that the Fourier Transform $H(u)$ is generally complex, and the complex conjugation is of vital significance to the operation.

This is again a linear operation, which is distributive, but however is not commutative, since if

$$c(x) = f(x) \otimes h(x)$$

then we can show that

$$h(x) \otimes f(x) = c(-x)$$

In two dimensions we have the correlation between two functions given by

$$c(x,y) = f(x,y) \otimes h(x,y) = \int \int f(s,t) h^*(s-x,t-y) ds \, dt$$

which in Fourier space gives,

$$C(u,v) = F(u,v) H^*(u,v)$$

(9)

Correlation is used in optics to characterise the incoherent optical properties of a system and in digital imaging as a measure of the “similarity” between two images.

### 6.1 Autocorrelation

If we consider the special case of correlation with two identical real space functions, we obtain the correlation of the input function with itself, being known as the Autocorrelation, being,

$$a(x,y) = f(x,y) \otimes f(x,y)$$

(11)

so that in Fourier space we have,

$$A(u,v) = F(u,v) F^*(u,v) = |F(u,v)|^2$$

(12)

which is the Power Spectrum of the function $f(x,y)$. Therefore the Autocorrelation of a function is given by the Inverse Fourier Transform of the Power Spectrum, giving,

$$a(x,y) = F^{-1}\{ |F(u,v)|^2 \}$$

(13)

In this case the correlation must be commutative, so we have that

$$a^*( -x, -y ) = a(x,y)$$

If in addition the function $f(x)$ is real, then clearly the correlation of a real function with itself is real, so that $a(x)$ is real. Therefore for a real function the autocorrelation is symmetric.