RSS Experiment Design

Model: \( y = f(x_1, x_2, \ldots, x_i, \ldots, x_n) \). \( y \) is dependent variable, \( x_i, i=1,\ldots,n \) are independent variables.

Independent variables specified by one-standard deviation tolerances \( \varepsilon_{x_i} \).

Nominal measurement of independent variables \( x_i, i=1,\ldots,n \)

Propagated uncertainty

\[
\varepsilon_y = \left( \left( \frac{\partial f}{\partial x_1} \right)_{x=x_i} \varepsilon_{x_1} \right)^2 + \left( \left( \frac{\partial f}{\partial x_2} \right)_{x=x_i} \varepsilon_{x_2} \right)^2 + \ldots + \left( \left( \frac{\partial f}{\partial x_n} \right)_{x=x_i} \varepsilon_{x_n} \right)^2 \right]^{1/2}
\]

Percent Contribution of Uncertainty as Design Tool

\[
\% \text{ Contribution from measurement } x_i = \frac{\left( \left( \frac{\partial f}{\partial x_i} \right)_{x=x_i} \varepsilon_{x_i} \right)^2}{\varepsilon_y^2} \times 100
\]

Structural Sensitivity

\[
\left| \frac{\partial f}{\partial x_i} \right|_{x=x_i}
\]

When Used: RSS is used when a mathematical model is available to relate dependent variables to independent variables.

Confidence Intervals

Confidence Interval on One Mean:

\[
\Pr \left\{ \bar{y} - t_{\alpha,v} \frac{s}{\sqrt{N}} < \mu < \bar{y} + t_{\alpha,v} \frac{s}{\sqrt{N}} \right\} = (1 - 2\alpha)100
\]

Degrees of Freedom \( v=N-1 \)

Sample Average

\[
\bar{y} = \frac{1}{N} \sum_{i=1}^{N} y_i
\]
Sample Standard Deviation
\[ s = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (y_i - \bar{y})^2} \]

Confidence Interval for Difference Between Two Means
\[
\Pr \left( (\bar{y}_1 - \bar{y}_2) - t_{\alpha,v} s_p \sqrt{\frac{1}{N_1} + \frac{1}{N_2}} < \mu_1 - \mu_2 < (\bar{y}_1 - \bar{y}_2) + t_{\alpha,v} s_p \sqrt{\frac{1}{N_1} + \frac{1}{N_2}} \right) = (1-2\alpha) 100
\]

Pooled Variance
\[ s^2_p = \frac{(N_1-1)s_1^2 + (N_2-1)s_2^2}{N_1 + N_2 - 2} \]

Degrees of Freedom \( v = N_1 + N_2 - 2 \)

When Used: Used when cause-and-effect is only determined by experimental data.

**Calibration/Regression**

Model: \( y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k \)

\( k \) Independent variables \( x_1, x_2, \ldots, x_k \). Considered known with infinite precision

Dependent variable \( y \), subject to measurement uncertainty.

\( n \) Data "Pairs" Assumed Measured. Data "Pair" = \((y, x_1, x_2, \ldots, x_k)\). Data placed into matrices

\[
\{y\} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad [X] = \begin{bmatrix} 1 & x_{11} & x_{21} & \cdots & x_{k1} \\ 1 & x_{12} & x_{22} & \cdots & x_{k2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{1n} & x_{2n} & \cdots & x_{kn} \end{bmatrix}
\]

Estimate for Fit Coefficients
\[ \hat{\beta} = [X^T X]^{-1} X^T y \]

Sum-Square Residuals
\[ SS_E = y^T y - \hat{\beta}^T X^T y \]
Sample Estimate of Residual Standard Deviation

\[ \hat{\sigma}^2 = \frac{SS_E}{n - p} \quad p = k + 1 \]

Covariance Matrix* \([C]=[X^TX]^{-1}\)

Confidence Interval for Fit Coefficients

\[ \Pr \left\{ \hat{\beta}_j - t_{\alpha,\upsilon} \sqrt{\hat{\sigma}^2 C_{jj}} < \beta_j < \hat{\beta}_j + t_{\alpha,\upsilon} \sqrt{\hat{\sigma}^2 C_{jj}} \right\} = (1-2\alpha)100 \]

Degrees of Freedom \(\nu = n - p\)

Prediction Estimate

\[ \hat{y} = x_o^T \hat{\beta} \]

where \(x_o = \{1 \ x_{o1} \ x_{o2} \ \ldots \ x_{ok}\}\).

Prediction Interval

\[ \hat{y} = x_o^T \hat{\beta}, \quad \Pr \left\{ \hat{y} - t_{\alpha,\upsilon} \sqrt{\hat{\sigma}^2 \left( 1 + x_o^T C x_o \right)} < y < \hat{y} + t_{\alpha,\upsilon} \sqrt{\hat{\sigma}^2 \left( 1 + x_o^T C x_o \right)} \right\} = (1-2\alpha)100 \]

Calibration Problem, Uncertainty on \(x\) given \(y\)

\[ \varepsilon_x = \left[ \left( \frac{1}{\hat{\beta}_1} \right)^2 \left( y - \hat{\beta}_0 \right)^2 + \left( \frac{1}{\hat{\beta}_1} \right)^2 \left( \frac{1}{\hat{\beta}_y} \right)^2 + \left( \frac{1}{\hat{\beta}_y} \right)^2 \right]^{1/2} \]

For tolerances, use

\[ \varepsilon_{\beta_0} = \frac{1}{2} \alpha, \upsilon \sqrt{\hat{\sigma}^2 C_{00}}, \quad \varepsilon_{\beta_1} = \frac{1}{2} \alpha, \upsilon \sqrt{\hat{\sigma}^2 C_{11}} \]

Regression used when a detailed numerical relationship between independent and dependent variables is desired.

* Our text denotes the quantity \(\hat{\sigma}^2 [C]\) as the covariance matrix.
Factorial Experiment Design and ANOVA

Design Grid; Factors \((A,B)\) , Levels \((a=2, b=2)\) and Replications \((n)\). Total number of experiments\(=n^{2^k}\)

Main Effects Factors

\[
\text{Main Effect A} = \left[\frac{ab-b + a-(1)}{n}\right]^{1/2}, \quad \text{Main Effect B} = \left[\frac{ab-a + b-(1)}{n}\right]^{1/2}
\]

Interactions Effect

\[
\text{Interaction Effect AB} = \left[\frac{ab-b - a-(1)}{n}\right]^{1/2}
\]

ANOVA Table and Interpretation

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>(a-1)</td>
<td>(MS_A)</td>
<td>(F=MS_A/MS_E)</td>
</tr>
<tr>
<td>(B)</td>
<td>(b-1)</td>
<td>(MS_B)</td>
<td>(F=MS_B/MS_E)</td>
</tr>
<tr>
<td>(AB)</td>
<td>((a-1)(b-1))</td>
<td>(MS_{AB})</td>
<td>(F=MS_{AB}/MS_E)</td>
</tr>
<tr>
<td>Error</td>
<td>(ab(n-1))</td>
<td>(MS_E)</td>
<td></td>
</tr>
</tbody>
</table>

If \(F>F_{\alpha,\nu_1,\nu_2}\), then probability is less than \(\alpha\cdot100\%\) that the observed \(F\) is due to statistical anomaly.

Factorial experiments are used when it is desired to learn of cause-and-effect relationships between independent variables and dependent variables using a minimum amount of data.