Calibration & Regression - Part II

ended with

$$\{\alpha\} = \{y\} - [X]\{\beta\}$$

find $\{\beta\}$ to minimize $\|\{\alpha\}\|_2 = \sum_{i=1}^{n} (\alpha_i)^2$

new variable

k - # of independant variables

previous case $k=1$ (only one variable x)

if $y = f(x, z, m, s) \ldots$ $k \neq 1$

Theorem

Given $n \times 1$ $\{y\}$ $n \times (k+1)$ $[X]$

then $(k+1) \times 1$ $\{\beta\}$ to minimize $\{\alpha\}$ is

$$\{\beta\} = \left[[X]^T[X]\right]^{-1} [X]^T \{y\}$$

gives $\{\beta\}$ in terms of known data points

when does $\{\alpha\} = 0$? when $n = 2$

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error on β

$$\{y\} = [X]\{\beta\} \rightarrow \{\beta\} = [X]^{-1}\{y\}$$

when using least squares formula it is only an estimate on $\{\beta\}$. place a hat on $\{\beta\}$

$$\{\hat{\beta}\} = \left[[X]^T[X]\right]^{-1}[X]^T\{y\}$$

$\wedge$ means estimate

└ is like a psudo inverse of $[X]$

when $n \to \infty$ $\{\hat{\beta}\} \to \{\beta\}$

Calibration ~~[illegible]~~ + Regression Part II

Tolerance (confidence interval) on $\{\beta\}$

Sum Square Error

$$SS_e = \{y\}^T\{y\} - 2\{\hat{\beta}\}^T[X]^T\{y\} + \{\hat{\beta}\}^T[X]^T[X]\{\hat{\beta}\}$$

like $\sum_{i=1}^{n}(y_i - \bar{y})^2$

$$\hat{\sigma}^2 = \frac{SS_e}{n-p}$$

$p = k+1$ $\quad$ k = # independent variables

n = # samples

Define $[C] = [[X]^T[X]]^{-1}$

diagonal most important part of $[C]$

$$\begin{bmatrix} C_{00} & & & \\ & C_{11} & & \\ & & C_{22} \cdots & \\ & & & C_{kk} \end{bmatrix}$$

so

$$Pr\left\{ \hat{\beta}_j - t_{\alpha,\nu}\sqrt{\hat{\sigma}^2 C_{jj}} < \beta_j < \hat{\beta}_j + t_{\alpha,\nu}\sqrt{\hat{\sigma}^2 C_{jj}} \right\} = (1-2\alpha)\cdot 100\%$$

j is an index $\quad$ $j=0$ - β_0 $\quad$ $j=1$ - β_1 ......

$\nu = n - p$ $\quad$ D.O.F.

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error on Y

Now, assume we will model

$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$

what is error on y?

Called a "prediction interval" on y

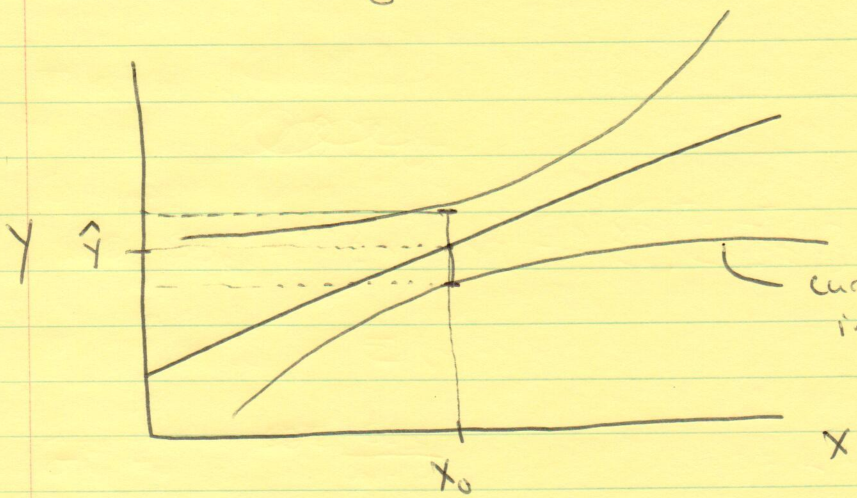
for $x = x_0$

$$Pr\left\{ \hat{y}(x_0) - t_{\alpha,\nu}\sqrt{\hat{\sigma}^2\left(1 + X_0^T[X^TX]^{-1}X_0\right)} < y < \hat{y}(x_0) + \text{same} \right\} = (1-2\alpha)\cdot 100$$

$\hat{y}(x_0) = \hat{\beta}_0 + \hat{\beta}_1 x_0$ $\quad$ $X_0 = \begin{Bmatrix} 1 \\ x_0 \end{Bmatrix}$

this is a confidence interval dependant on <u>where</u> we chose to make measurement

Calibration + Regression - Part II

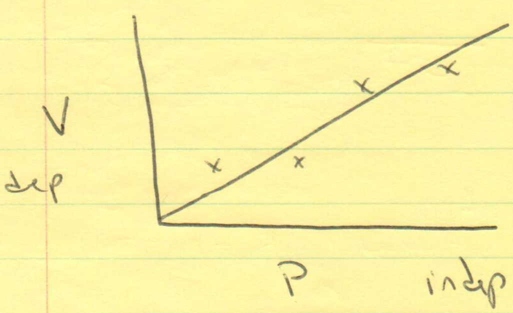


curve because of variation in β_1 (exagerated)

Calibration problem

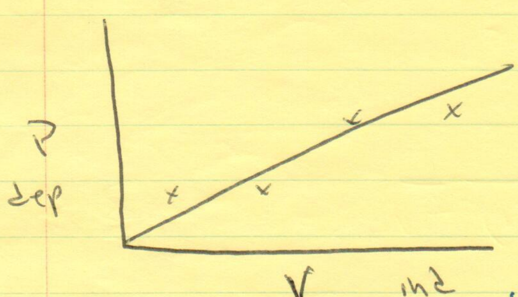
P → Transducer → V

input pressure + get out voltage
↑
known



$$V = \beta_0 + \beta_1 P$$

After calibration - when using (regression)



Know V

$$\underbrace{P = \frac{V - \beta_0}{\beta_1}}_{\text{specific case}} \Rightarrow \underbrace{\boxed{x = \frac{y - \beta_0}{\beta_1}}}_{\text{general case}}$$

Calibration & Regression Part II

We seek a confidence interval on P, but have confidence on β_0, β_1, and error spec ε_V on V

Use RSS method to get confidence interval

$$\varepsilon_x = \sqrt{\left(\frac{\partial x}{\partial y}\varepsilon_y\right)^2 + \left(\frac{\partial x}{\partial \beta_0}\varepsilon_{\beta_0}\right)^2 + \left(\frac{\partial x}{\partial \beta_1}\varepsilon_{\beta_1}\right)^2}$$

$\frac{1}{\beta_1}$ 1 std dev on volt measurement $\frac{-1}{\beta_1}$ 1 std dev on β_0 $-\frac{(y-\beta_0)}{\beta_1^2}$

$t(\alpha,\nu)\sqrt{\hat{\sigma}^2 C_{jj}}$ for $(1-2\alpha)\cdot 100\% = 67\%$

note ε_x depends on values of β_0, β_1, error on β_0 + β_1 + ε_V and actual value of y

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Have all tools to do HW now
Look @ HW

assume $\varepsilon_{voltmeter} = 0.005V$
for part 2, assume y = mean value of y