1. The Joy of Sets

Just as musicians must first learn to play individual notes on their instruments, budding mathematicians must first become adept at working with sets. This leads to our first definition:

Definition 1.1. A set is a collection of objects.

We usually denote sets by capital letters, and we can describe sets in a number of ways. For example:

- \( A := \{1, 2, 7\} \) is a set with three members or elements in it. We use the notation “1 ∈ A” to denote that 1 is an element of A. On the other hand, 47 ∉ A.
- \( \mathbb{N} := \{1, 2, 3, \ldots\} \) denotes the set of all positive integers and is known as the set of natural numbers. In this case, 47 ∈ N but π ∉ N.
- \( B := \{x : 6x^5 - 27x^2 + 433x + \pi = 0\} \) is a set whose members are well defined but difficult to determine explicitly. When converting this definition to English, you should read it as “B is the set of all x such that 6x^5 - 27x^2 + 433x + \pi = 0.”

In all of these examples, we use the symbol “:=” to indicate that we are defining something (namely, the sets A, N, and B, respectively). The set \( \mathbb{N} \) shows up often, as do lots of other special sets with their own symbols, such as the empty set:

Definition 1.2. The empty set is the set with no elements and is denoted by \( \emptyset \).

Some other good sets to know are

- \( \mathbb{Z} := \{0, 1, -1, 2, -2, \ldots\} \), the set of all integers,
- \( \mathbb{Q} := \{\frac{n}{m} : n \in \mathbb{Z}, m \in \mathbb{Z}, m \neq 0\} \), the set of rational numbers, and
- \( \mathbb{R} \), the set of all real numbers.

Although these are some pretty interesting examples, we rarely focus on just one or two particular sets. Instead, we often start with a few sets of interest and then mash them into what we want using the set operations defined next. In these definitions, \( A \) and \( B \) are arbitrary sets.

Definition 1.3. \( A \) is a subset of \( B \), denoted \( A \subset B \), if every element of \( A \) is an element of \( B \). In other words, \( A \subset B \) means that \( x \in B \) whenever \( x \in A \).

For example, observe that \( \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \).

Definition 1.4. \( A \) and \( B \) are equal, denoted \( A = B \), if they have the same elements. Thus, \( A = B \) if and only if \( A \subset B \) and \( B \subset A \).
**Definition 1.5.** The *union* of *A* and *B*, denoted *A* ∪ *B*, is the set consisting of elements of either *A* or *B*. (Here, “or” is meant inclusively.) Thus,

\[ A \cup B := \{ x : x \in A \text{ or } x \in B \}. \]

**Definition 1.6.** The *intersection* of *A* and *B*, denoted *A* ∩ *B*, is the set consisting of those elements belonging to both *A* and *B*. Thus,

\[ A \cap B := \{ x : x \in A \text{ and } x \in B \}. \]

**Definition 1.7.** The *difference* between sets *A* and *B*, denoted *A* \* *B*, is the set of elements in *A* which are not in *B*. Thus,

\[ A \setminus B := \{ x : x \in A \text{ and } x \notin B \}. \]

Many of these operations can be represented visually with Venn diagrams. These are very useful for developing intuition, but a Venn diagram is NOT a proof! In any case, you should do

**Problem 1.1.** Let *X* be a universal set, and let *A*, *B*, and *C* be subsets of *X*. Draw Venn diagrams representing the following:

1. \( X \setminus A \)
2. \( A \cap (B \cup C) \)
3. \( (X \setminus A) \cap (X \setminus B) \)
4. \( (A \setminus B) \cup (B \setminus A) \) (This operation actually has a name: the resulting set is the *symmetric difference* of *A* and *B*, denoted \( A \Delta B \).)

From now on, you will often see problems that are stated as if they were true. Your first job is to determine if the given statement is indeed true. If it is true, provide a proof; otherwise, you must provide a counterexample verifying that the statement is false.

**Problem 1.2.** For any set *A*, \( \emptyset \subset A \).

**Problem 1.3.** If *A* ⊂ *B*, then *A* ∩ *C* ⊂ *B* ∩ *C*.

**Problem 1.4.** *A* \* *B* = *B* \* *A*

**Problem 1.5.** *A* ⊂ *B* if and only if *A* ∪ *B* = *B*.

**Problem 1.6.** *A* ∩ (*B* ∪ *C*) = (*A* ∩ *B*) ∪ (*A* ∩ *C*)

**Problem 1.7.** *A* ∩ *B* = *A* ∩ (*B* \* *A*)

**Problem 1.8.** *A* ∪ (*B* ∩ *C*) = (*A* ∪ *B*) ∩ (*A* ∪ *C*)

**Problem 1.9.** *A* ∪ *B* = *A* ∪ (*B* \* *A*)

**Problem 1.10.** Let *X* be a universal set, with *A* ⊂ *X* and *B* ⊂ *X*. Then

\[ X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) . \]
The universal set $X$ in question is usually understood if not given explicitly. Sets like $X \setminus A$ and $X \setminus B$ occur frequently enough to have a name:

**Definition 1.8.** The **complement** of a set $A$, denoted $A^c$, is the set of elements in $X$ which are not in $A$. Thus,

$$A^c := \{ x : x \notin A \}.$$ 

Sometimes we need to consider lots of sets at the same time. For example, suppose that $A_1, A_2, \ldots, A_n$ are sets, where $n$ is some natural number. Then

$$\bigcap_{i=1}^{n} A_i := \{ x : x \in A_i \text{ for each } i, \ 1 \leq i \leq n \}.$$ 

and

$$\bigcup_{i=1}^{n} A_i := \{ x : x \in A_i \text{ for some } i, \ 1 \leq i \leq n \}.$$ 

More generally, we can use an index set $I$ to describe a family of sets. Thus, we might write $B := \{ A_i \}_{i \in I}$ to define the set $B$ whose elements are the sets $A_i$ for each $i \in I$. We could then consider the union

$$\bigcup_{i \in I} A_i := \{ x : x \in A_i \text{ for some } i \}$$

and the intersection

$$\bigcap_{i \in I} A_i := \{ x : x \in A_i \text{ for all } i \}.$$ 

**Problem 1.11.**

$$\left( \bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c$$

**Problem 1.12.**

$$\left( \bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c$$

**Definition 1.9.** The **power set** of a set $A$, denoted $\mathcal{P}(A)$, is the set of all subsets of $A$. Thus,

$$\mathcal{P}(A) := \{ B : B \subseteq A \}.$$ 

**Problem 1.13.** How many elements are in $\mathcal{P}(A)$ if $A := \{1, 2, 3, \ldots, 10\}$?

**Problem 1.14.** Let $n \in \mathbb{N}$. How many elements are in $\mathcal{P}(A)$ if $A$ has $n$ elements?
We usually take the notion of a set for granted, but this is a good time to point out that problems arise if we aren’t careful. The power set $\mathcal{P}(A)$ of $A$ is an example of a set whose elements are themselves sets, and this might make you wonder about the possibility of a set $A$ that belongs to itself. To that end, consider

**Definition 1.10.** A set $A$ is **abnormal** if $A \in A$. $A$ is **normal** if it is not abnormal.

Now let $\mathcal{N}$ denote the set of all normal sets,

$$\mathcal{N} := \{ A : A \text{ is normal} \}.$$

**Problem 1.15.**

1. If $\mathcal{N}$ is normal, then $\mathcal{N} \in \mathcal{N}$.
2. If $\mathcal{N}$ is abnormal, then $\mathcal{N} \notin \mathcal{N}$.
3. Is $\mathcal{N}$ normal or abnormal?

Mathematicians like to rule out irritations like this, so most of them accept axioms (the Zermelo–Fraenkel axioms of set theory) that prevent such things from popping up. Although we won’t worry about this any more, you might want to investigate mathematical logic on your own if you want to learn more.

**Definition 1.11.** A nonempty set $A$ is **finite** if there is a one-to-one correspondence between the elements of $A$ and the elements of the set $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. The empty set is also finite.

**Definition 1.12.** A set $A$ is **infinite** if it is not finite.

**Definition 1.13.** A set $A$ is **countably infinite** if there is a one-to-one correspondence between the elements of $A$ and the elements of $\mathbb{N}$.

**Definition 1.14.** A set $A$ is **uncountable** if it is neither finite nor countably infinite.

**Definition 1.15.** A set $A$ is **countable** if it is either finite or countably infinite.

**Problem 1.16.** Show that $\mathbb{Z}$ is countable.

**Problem 1.17.** Show that $\mathbb{Q}$ is countable.

**Problem 1.18.** We discussed one way of enumerating the rationals in class. Using this basic idea to enumerate the **positive** rationals, how long do we have to wait for the positive rational number $m/n$ to appear in our list? In other words, determine a number $N$ so that you can guarantee that this number will appear within the first $N$ terms of our resulting list of positive rationals. (Don’t try to be exact! Just get a rough estimate for $N$.)

**Problem 1.19.** Is $\mathbb{R}$ countable?
Some further comments on the set $\mathbb{R}$ are in order. We defined the rationals $\mathbb{Q} \subset \mathbb{R}$ above; the \textit{irrational numbers} are those real numbers which are not rational. For example,

\textbf{Problem 1.20.} $\sqrt{2}$ is irrational.

The reals are a well-mixed blend of rationals and irrationals. In fact, the following hold:

- Suppose that $a$ and $b$ are real numbers with $a < b$. Then there is another real number $c$ such that $a < c < b$.
- Suppose that $a$ and $b$ are rational numbers with $a < b$. Then there is an irrational number $c$ such that $a < c < b$.
- Suppose that $a$ and $b$ are irrational numbers with $a < b$. Then there is a rational number $c$ such that $a < c < b$.

\section{2. Getting to know your neighborhood}

\textbf{Definition 2.1.} The \textit{plane} $\mathbb{R}^2$ is the set of all ordered pairs of real numbers,

$$\mathbb{R}^2 := \{(x, y) : x, y \in \mathbb{R}\}.$$ 

More generally, for a positive integer $n \in \mathbb{N}$ we define $\mathbb{R}^n$ to be the set of all ordered $n$–tuples,

$$\mathbb{R}^n := \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R} \text{ for all } i\}.$$ 

\textbf{Definition 2.2.} If $p = (p_1, p_2)$ and $q = (q_1, q_2)$ are points in the plane, then the \textit{distance} between $p$ and $q$ is

$$d(p, q) := \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}.$$ 

More generally, if $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ are points in $\mathbb{R}^n$, then the \textit{distance} between $p$ and $q$ is

$$d(p, q) := \sqrt{n \sum_{i=1}^{n} (p_i - q_i)^2}.$$ 

\textbf{Definition 2.3.} If $p = (p_1, p_2)$ and $q = (q_1, q_2)$ are points in the plane, then the \textit{dot product} of $p$ and $q$, denoted $p \cdot q$, is the number

$$p \cdot q := p_1 q_1 + p_2 q_2.$$ 

More generally, if $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ are points in $\mathbb{R}^n$, then their \textit{dot product} is the number

$$p \cdot q := \sum_{i=1}^{n} p_i q_i.$$
Observing that
\[ d(p, q) = \sqrt{(p - q) \cdot (p - q)} \]
leads to

**Definition 2.4.** The **norm** or **length** of \( p \in \mathbb{R}^n \) is the number
\[ |p| := d(p, 0) = \sqrt{p \cdot p} . \]

**Problem 2.1.** Prove the following results related to the distance function and the notion of length just defined.

1. For any two points \( p, q \in \mathbb{R}^n \), \( d(p, q) \geq 0 \). Moreover, \( d(p, q) = 0 \) if and only if \( p = q \).
2. Prove the **Cauchy-Schwarz inequality**: for points \( p, q \in \mathbb{R}^n \),
\[ p \cdot q \leq |p| |q| . \]
Hint: First compute \( (p - q) \cdot (p - q) \) for points \( p \) and \( q \) such that \( |p| = |q| = 1 \).
Then, for the general case, start with arbitrary \( p \) and \( q \) and turn them into vectors of length one.
3. Use the Cauchy-Schwarz inequality to prove the **triangle inequality**: for points \( p, q \in \mathbb{R}^n \),
\[ |p + q| \leq |p| + |q| . \]
Hint: Start by computing \( (p + q) \cdot (p + q) \).
4. Use the previous version of the triangle inequality to prove its alternative formulation: for points \( p, q, r \in \mathbb{R}^n \),
\[ d(p, q) \leq d(p, r) + d(r, q) . \]

**Definition 2.5.** If \( p_0 \in \mathbb{R}^n \) and \( \varepsilon > 0 \), then the **spherical neighborhood about** \( p_0 \) **of radius** \( \varepsilon \) is the set
\[ B(p_0, \varepsilon) := \{ p \in \mathbb{R}^n : d(p, p_0) < \varepsilon \} . \]

To take a break from all of these definitions, do

**Problem 2.2.** Sketch the following subset of the plane:
\[ B((2, 2), 1) \cup B((4, 2), 1) \cup B((3, 1), \sqrt{2}) \]

**Definition 2.6.** A set \( A \subset \mathbb{R}^n \) is **open** if every point \( p \in A \) has a spherical neighborhood contained entirely in \( A \). In other words, \( A \) is open if, for each \( p \in A \), there is some \( \varepsilon > 0 \) such that
\[ B(p, \varepsilon) \subset A . \]

**Note.** The radius \( \varepsilon > 0 \) in the previous definition will usually be different for different points \( p \) in the open set \( A \!).
Problem 2.3. For any point $p$ and any given $\varepsilon > 0$, the spherical neighborhood $B(p, \varepsilon)$ is an open set.

Problem 2.4. If $A$ and $B$ are open sets, then both $A \cup B$ and $A \cap B$ are open.

Definition 2.7. A set $A \subseteq \mathbb{R}^n$ is **closed** if its complement $A^c$ is open.

Problem 2.5. If $A$ and $B$ are closed sets, then both $A \cup B$ and $A \cap B$ are closed.

Problem 2.6. Determine whether the following subsets of $\mathbb{R}^2$ are open, closed, both, or neither.

1. $\{(x, y) : x, y > 0\}$
2. $\{(x, y) : x^2 + y^2 < 4 \} \setminus \{(x, y) : x^2 + y^2 < 1\}$
3. $\{(x, y) : x^2 = y^2\}$
4. $\mathbb{R}^2$
5. $\{(x, y) : x, y \in \mathbb{Z}\}$
6. $\{(x, y) : |x| < y\}$
7. $\{(x, y) : \frac{1}{x}, \frac{1}{y} \in \mathbb{Z}\}$
8. $\{(x, y) : x^2 + y^2 < 1\} \cup \{(2, 2)\}$

Problem 2.7. Let $I$ be an index set, and suppose that $A_i$ is an open set for every $i \in I$.

1. $\bigcup_{i \in I} A_i$ is open.
2. $\bigcap_{i \in I} A_i$ is open.

Problem 2.8. Let $I$ be an index set, and suppose that $A_i$ is a closed set for every $i \in I$. What does the previous problem tell us about the union and intersection of these sets?

Definition 2.8. The point $p$ is a **cluster point** for the set $A$ if every spherical neighborhood about $p$ contains infinitely many elements of $A$.

Definition 2.9. The **closure** of the set $A$, denoted $\overline{A}$, is the union of $A$ and all of its cluster points.

Definition 2.10. The point $p$ is an **isolated point** for the set $A$ if there is a spherical neighborhood $B(p, \varepsilon)$ such that $B(p, \varepsilon) \cap A = \{ p \}$.
Definition 2.11. The point $p$ is a *boundary point* for the set $A$ if every spherical neighborhood about $p$ intersects both $A$ and $A^c$. The *boundary* of $A$, denoted $\partial A$, is the set of all boundary points of $A$.

Problem 2.9. For each of the subsets of $\mathbb{R}^2$ defined in Problem 2.6, determine the cluster points of the set, the closure of the set, the isolated points of the set, and the boundary of the set.

Problem 2.10. A closed set contains all of its cluster points.

Problem 2.11. A set with infinitely many elements has at least one cluster point.