Homework 5

MATH 472

1) Please email me your homework as a single pdf file.
2) Show your work clearly. Justify all your answers.

1. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Prove that

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2}{4}$$

   (This identity is called the polarization identity.)

2. The points $\mathbf{u}_1, \dots, \mathbf{u}_k$ in $\mathbb{R}^n$ are said to be an *orthonormal* set if $\|\mathbf{u}_i\| = 1$ for $1 \le i \le k$ and $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ if $i \neq j$, $1 \le i, j \le k$. Suppose that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthonormal set. For $\mathbf{u} = \alpha_1 \mathbf{u}_1 + \cdots + \alpha_k \mathbf{u}_k$, show that

$$\|\mathbf{u}\| = \sqrt{\sum_{i=1}^{k} \alpha_i^2}.$$

3. Find the limit of the sequence $\{(\frac{(\sin n)^n}{n}, \frac{1}{n^2})\}_{n=1}^{\infty}$ in $\mathbb{R}^2$.

4. Find the interior, i.e., $\mathrm{int}(S)$, of the set

$$S = \{(x, y, z) \in \mathbb{R}^3 : 0 \le x < 1, y^2 + z^2 \le 1\}.$$

5. Is $S = \{(x, y) \in \mathbb{R}^2 : x, y \ge 1\}$ closed? Why, or why not?

6. Let m and n be natural numbers. Find a necessary condition on m and n such that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^n y^m}{x^2 + y^2}$$

   exists.

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7. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} (x/|y|)\sqrt{x^2+y^2} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

Show that f is not continuous at $(0,0)$.
Hint: Consider the sequence of points $\{(\frac{1}{k}, \frac{1}{k^2})\}$. Note that this sequence converges to $(0,0)$. Show that the sequence $\{f(\frac{1}{k}, \frac{1}{k^2})\}$ does not converge to $f(0,0)$.